D-MATH
FS 2020
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Solutions 1

Algebraic Geometry

1 Let $J = (X_1^3 - X_2^6, X_1X_2 - X_2^3)$. By the Hilbert's Nullstellensatz,

$$\sqrt{J} = I(V(J)).$$

One can easily check that

$$V(J) = \{(a^2, a) : a \in \mathbb{C}\},$$

so

$$I(V(J)) = \{p \in \mathbb{C}[X_1, X_2] : p(a^2, a) = 0 \ \forall a \in \mathbb{C}\}.$$

This implies in particular that $p(X_2^2, X_2) = 0$, hence $I(V(J)) \subseteq (X_1 - X_2^2)$. On the other hand, $X_1 - X_2^2 \in I(V(J))$, so $\sqrt{J} = (X_1 - X_2^2)$.

2 Let $A(X) = K[X_1, \ldots, X_n]/I(X)$. Then $A(X)$ is a field if and only if the ideal $I(X)$ is maximal. By the Weak Nullstellensatz, $I(X)$ must be of the form

$$I(X) = (X_1 - a_1, \ldots, X_n - a_n)$$

for some $a := (a_1, \ldots, a_n) \in \mathbb{A}^n$. Then $V(I(X)) = X = \{a\}$.

3 See Remark 1.14 of Gathmann-KÃ¼hn notes.

4 One always has $X \subseteq V(I(X))$. Since $V(I(X))$ is closed, one also gets $\overline{X} \subseteq V(I(X))$. Recall that the closure of X is the set of points $a \in \mathbb{A}^n$ so that every neighborhood of a intersects X. The neighborhoods of a in the Zaroski topology are of the form $\mathbb{A}^n - V(J)$ for some ideal J with $a \notin V(J)$. If $a \in V(I(X))$, then by definition $f(a) = 0$ for all polynomial f vanishing on X. Let now $g \in K[X_1, \ldots, X_n]$ with $g(a) \neq 0$. By the above, there exists a $x \in X$ such that $g(x) \neq 0$, so $x \notin V(g)$ and $x \in X$.

5 By substituing $X_1 = X_2X_3$ to the other equation one gets

$$X_2(X_3^2 - X_2) = 0,$$

so the two irreducible components are $V(X_1, X_2)$ and $V(X_1 - X_3^2, X_2 - X_3^2)$.

(6) This is a basic exercise of topology.

(7) Let

$$A_0 \subset A_1 \subset \cdots \subset A_n$$

be a chain of irreducible closed subsets of A. Write $A_i = C_i \cap A$ for some $C_i \subseteq X$ closed. By taking the closures, one gets a chain of irreducible closed subsets of X

$$\overline{A_0} \subseteq \cdots \subseteq \overline{A_n}.$$

It remains to show that the inclusions are strict. For all $i = 0, \dots, n$, pick $x \in A_{i+1} - C_i$. Then $x \notin \overline{A_i} \subseteq C_i$.

(8)

$$X = \left\{ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} : \text{rank}\, A \leq 1 \right\} \subseteq \mathbb{A}^6.$$

This is equivalent to say that all the 2×2 minors are zero, i.e. X is the zero locus of the ideal

$$(X_{11}X_{22} - X_{21}X_{12}, X_{11}X_{23} - X_{21}X_{13}, X_{12}X_{23} - X_{22}X_{13}).$$

(9) Assume that I can be generated by two polynomials, say $I = (f, g)$. Since I is a monomial ideal, one has that the monomials of f and g belong to I. This implies that both f and g have no degree ≤ 1 monomials. Also, one has that $X_1X_2|f_2$ or $X_1X_3|f_2$ or $X_2X_3|f_2$, where f_2 is the degree-2 monomial part of f. Same holds for g. Assume that $X_1X_2|f_2$ and $X_2X_3|g_2$, then the monomial X_1X_3 does not appear in any polynomial combination $\alpha f + \beta g$ ($\alpha, \beta \in \mathbb{C}[X_1, X_2, X_3]$). Hence $X_1X_3 \notin (f, g)$.