D-MATH
FS 2020
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Solutions 2

Algebraic Geometry

1 See Remark 2.31.a of Gathmann's notes.

2 Being noetherian is equivalent to the property that every non-empty family of open subsets has a maximal element. Consider an open cover $\bigcup_{i\in I} U_i = X$ of X and the family

$$\mathcal{F} = \{\bigcup_{i\in I_0} U_i : I_0 \subseteq I \text{ finite}\}.$$

Then $\mathcal{F}$ has a maximal element $\bigcup_{i\in I_0} U_i$. Assume $\bigcup_{i\in I_0} U_i \subset X$ and pick $x \in X - \bigcup_{i\in I_0} U_i$. But $X = \bigcup_{i\in I} U_i$, so there exists $i \in I - I_0$ such that $x \in U_i$. Hence $U_i \cup \bigcup_{i\in I_0} U_i \supset \bigcup_{i\in I_0} U_i$, which contradicts the maximality of $\bigcup_{i\in I_0} U_i$.

3

a. Let $a \in U$; $(U,\phi) \sim (U,\psi)$ means that there exists $V_a \subseteq U$ such that $\phi_{|V_a} = \psi_{|V_a}$. Clearly $\{V_a\}_{a\in U}$ is an open cover of U such that $\phi_{|V_a\cap V_b} = \psi_{|V_a\cap V_b}$. By the gluing property there exists a unique $\xi \in \mathcal{F}(U)$ such that $\xi_{|V_a} = \phi_{|V_a} = \psi_{|V_a}$ for all $a \in U$. In particular both ϕ and ψ do the job, by uniqueness we conclude.

b. Consider the zero lucus $V(\phi - \psi)$, where ϕ and ψ agree on a neighborhood $V_a \subseteq U$ of $a \in U$. Then $V_a \subseteq V(\phi - \psi)$ and $V(\phi - \psi)$ is closed in U. Hence $\overline{V_a}^U \subseteq V(\phi - \psi)$. Also, U is irreducible (since X is), then $\overline{V_a}^U = U$.

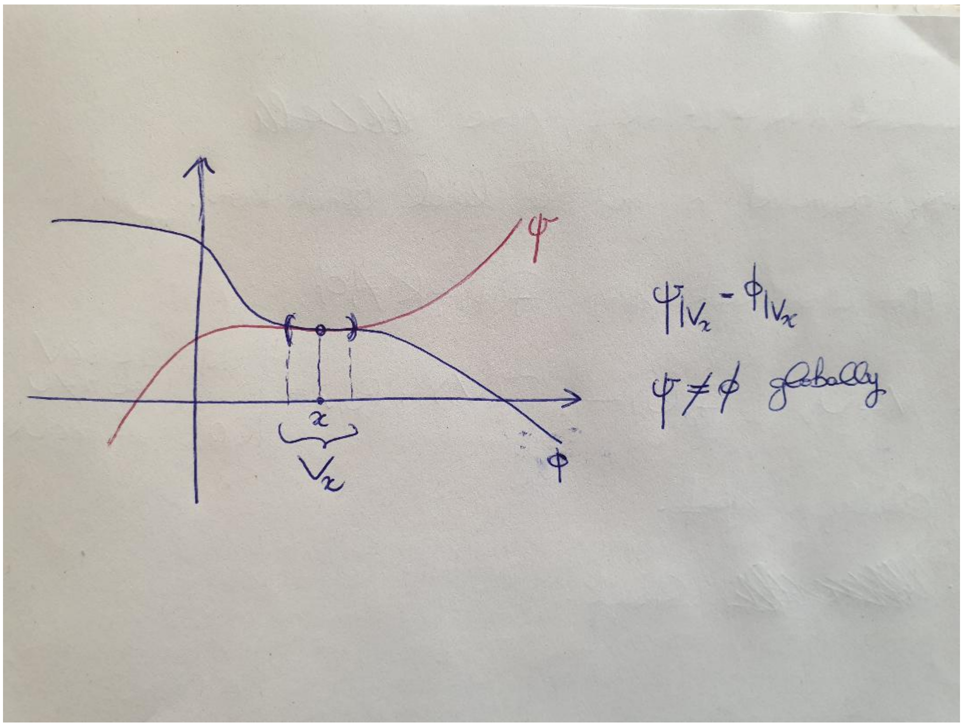
c. Consider ψ and ϕ two functions looking like the ones in the figure below. Let you the pleasure to write down an explicit expression for them.

4

a. By the homomorphism $\mathcal{F}_a \twoheadrightarrow \mathbb{R}$, $\phi \mapsto \phi(a)$ we see that the kernel I_a is maximal, since $\mathcal{F}_a/I_a \simeq \mathbb{R}$. It is the only maximal ideal, since $\mathcal{F}_a - I_a = \mathcal{F}_a^\times$:
If $\phi(a) \neq 0$, then if $U' := X - \{\phi = 0\}$,

$$(U,\phi)\cdot(U', 1/\phi) \sim (U\cap U', 1)$$

in $\mathcal{F}_a$, since $a \in U \cap U'$.



b. Consider $(\mathbb{R}, X-1) \in \mathcal{F}_0$. Then $X - 1_{|0} \neq 0$, but $(\mathbb{R}, X-1)$ is not a unit in $\mathcal{F}_0^\times$: if ψ defined on $0 \in U$ is such that

$$\psi \cdot (X-1) = 1$$

on U, then ψ cannot be a polynomial on U.

5 If $\mathrm{codim} Y = 1$, then $I(Y) = (f)$ for some $f \in A(X)$ irreducible. Then

$$\mathcal{O}_X(D(f)) \simeq A(X)_f.$$

Assume $\mathrm{codim} Y \geq 2$ and let $I(Y) = (f_1, \ldots, f_k)$ $(k \geq 2)$. Since $X - Y = \cup_{i=1}^k D(f_i)$ and $\mathcal{O}_X(D(f_i)) \simeq A(X)_{f_i}$, one has $\phi = g_i/f_i^{m_i}$ on $D(f_i)$ for some $g_i \in A(X)$, $m_i \in \mathbb{N}$ with $f_i \nmid g_i$. On $D(f_i) \cap D(f_j)$ $(i \neq j)$, $g_i f_j^{m_j} = g_j f_i^{m_i}$. This also holds on $\overline{D(f_i) \cap D(f_j)} = X$. Since $A(X)$ is a UFD, $m_i = 0$, otherwise $f_i | g_i$. Therefore $\phi = g_i$, a polynomial on $D(f_i)$ for all i.

6 a. It's a sheaf, the reason is that being continuous is a local condition.

b. It's not a sheaf, since being bounded is a global condition.

c. Not even a presheaf, since $\mathcal{F}(\emptyset) = K \neq 0$.

d. If $V \subseteq U$,

$$\rho_{U,V} = \begin{cases} \mathrm{id} & \text{if } x \in V \text{ or } x \notin U \\ 0 & \text{if } x \in U - V. \end{cases}$$

It's clearly a sheaf.

e. In the category of rings for instance, it's not a presheaf: $\mathcal{F}(\emptyset) \neq 0$.