EXERCISES ON ALGEBRAIC GROUPS

1. Exercises

(1) Onischik–Vinberg, Problem 3.1.2.
(2) Onischik–Vinberg, Problem 3.1.3.
(3) Onischik–Vinberg, Problem 3.1.4.
(4) Onischik–Vinberg, Problem 3.1.7.
(5) Onischik–Vinberg, Problem 3.1.8.
(6) Milne, Problem 3-1.

optional Let G be an algebraic group. Let $A := k[G]$ denote its coordinate ring, equipped with the right regular representation r_A of G. Let $g \in G$. Show that g is semisimple (resp. unipotent) if and only if $r_A(g)$ has semisimple (resp. unipotent) restriction to each finite-dimensional subrepresentation of A. (Use the proof technique given in lecture of the existence and uniqueness of Jordan decompositions.)

(7) Onishchik–Vinberg, Problem 3.2.1.
(8) Onishchik–Vinberg, Problem 3.2.2.
(9) Suppose k has chacteristic zero. Show that each $g \in \mathrm{GL}(V)$ of finite order (i.e., $g^n = 1$ for some natural number n) is semisimple. Show also that if g^n is semisimple for some natural number n, then g is semisimple. Show by example that either implication may fail if k has positive characteristic.
(10) Show that every connected diagonalizable algebraic group is a torus.
(11) Let H be an algebraic subgroup of a torus T. Let $H^\perp := \{\chi \in \mathfrak{X}(T) : \chi|_H = 1\}$. Show that $H = \{t \in T : \chi(t) = 1 \text{ for all } \chi \in H^\perp\}$.
(12) Let G be an algebraic group. For $g \in G$, write $G(g)$ for the Zariski closure of the group $\{g^n : n \in \mathbb{Z}\}$ generated by g. Verify (using an earlier homework problem) that $G(g)$ is a commutative algebraic group. Show that $G(g)_s = G(g_s)$ and $G(g)_u = G(g_u)$. Deduce that the multiplication map defines an isomorphism of commutative algebraic groups $G(g) \cong G(g_s) \times G(g_u)$.
(13) Using Theorem 11.2 and Lemma 12.2 in the course synopsis, prove the following: for any connected nilpotent algebraic group G, the subsets G_s and G_u are normal algebraic subgroups, and the multiplication map $G_s \times G_u \to G$ is an isomorphism of algebraic groups.
(14) (a) Write down a careful proof (following, e.g., the course references) that the image of the Grassmannian $G_{n,m}$ under the Plücker embedding $G_{n,m} \hookrightarrow \mathbb{P}^N = \mathbb{P}(\Lambda^m(k^n))$ is closed.

(b) Show that GL_n is irreducible.
(c) Show that the action of GL_n on $G_{n,m}$ is transitive.
(d) Verify that GL_n acts algebraically on $\mathbb{P}^N$, hence continuously on $G_{n,m}$; deduce that the Plücker image of $G_{n,m}$ is irreducible.

Solution 6: Let Γ be a group. Let $R(\Gamma)$ be the k-algebra of maps $\Gamma \to k$ on Γ. Note that $R(\Gamma) \otimes R(\Gamma)$ acts on $\Gamma \times \Gamma$ by $f \otimes h(g_1, g_2) := f(g_1)h(g_2)$. Recall the maps defining Hopf algebra

$$\Delta : R(\Gamma) \to R(\Gamma \times \Gamma), \quad \Delta f(g_1, g_2) = f(g_1 g_2),$$

$$\epsilon : R(\Gamma) \to k, \quad \epsilon f(g) = f(1),$$

$$S : R(\Gamma) \to R(\Gamma), \quad S f(g) = f(g^{-1}).$$

Let $\rho_n : \Gamma \to \mathrm{GL}_n(k)$ be n-dimensional representations of Γ. Consider the family of functions $\tilde{\rho}_{n,i,j} : g \mapsto \rho_n(g)_{i,j}$ on Γ. Let $R'(\Gamma)$ be the subspace of $R(\Gamma)$ spanned by the $\tilde{\rho}_{n,i,j}$ for various n, i, j. $\tilde{\rho}$ are called *matrix coefficients* of the representation ρ and $R'(\Gamma)$ is the space of (finite dimensional) matrix coefficients of Γ.

- We need to show that $R'(\Gamma)$ is a subalgebra of $R(\Gamma)$. That is, we need to show that

  $$\tilde{\rho}_{n,i,j}\tilde{\rho}_{m,k,l} \in R'(\Gamma),$$

  for $i, j \leq n$ and $k, l \leq m$. We introduce the natural Γ-invariant Euclidean inner product $\langle , \rangle$ on ρ_n. Then

  $$\tilde{\rho}_{n,i,j}(g) = \langle \rho_n(g)e_i, e_j \rangle,$$

  where $\{e_i\}_i$ is the standard basis of k^n. Consider the representation $\rho_n \otimes \rho_m$ as diagonal representation of Γ, which is a finite dimensional representation of Γ. Consider the matrix coefficient

  $$\langle \rho_n \otimes \rho_m(g)e_i \otimes e_k, e_j \otimes e_l \rangle \in R'(\Gamma).$$

  The above equals to $\langle \rho_m(g)e_i, e_j \rangle \langle \rho_n(g)e_k, e_l \rangle = \tilde{\rho}_{n,i,j}\tilde{\rho}_{m,k,l}(g)$.
- We need to show that Δ maps $R'(\Gamma)$ to $R'(\Gamma) \otimes R'(\Gamma)$. A priori, $\Delta(R'(\Gamma))$ is inside $R'(\Gamma \times \Gamma)$ given by

  $$\Delta(\tilde{\rho}_{n,i,j})(g_1, g_2) = \tilde{\rho}_{n,i,j}(g_1 g_2).$$

  The RHS of the above is the i, j'th matrix coefficient of $g_1 g_2$, which can also be written as

  $$\sum_{k=1}^{n} \tilde{\rho}_{n,i,k}(g_1)\tilde{\rho}_{n,k,j}(g_2) = \sum_{k=1}^{n} \tilde{\rho}_{n,i,k} \otimes \tilde{\rho}_{n,k,j}(g_1, g_2).$$

  Hence the above sum lies in $R'(\Gamma) \otimes R'(\Gamma)$.
- We need to show that Δ, ϵ, S define a Hopf algebra structure on $R'(\Gamma)$. That is, we need to show the following relations:
  (1)

  $$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta.$$

To show this let $g_i \in \Gamma$ for $i = 1, 2, 3$. Then using the previous problem we obtain

$$\begin{aligned}
&(\mathrm{id} \otimes \Delta)(\Delta(\tilde{\rho}_{n,i,j}))(g_1, g_2, g_3) \\
&= (\mathrm{id} \otimes \Delta)(\sum_k \tilde{\rho}_{n,i,k} \otimes \tilde{\rho}_{n,k,j})(g_1, g_2, g_3) \\
&= \sum_k \tilde{\rho}_{n,i,k} \otimes \Delta(\tilde{\rho}_{n,k,j})(g_1, g_2, g_3) \\
&= \sum_k \tilde{\rho}_{n,i,j}(g_1)\tilde{\rho}_{n,k,j}(g_2 g_3) \\
&= \sum_k \tilde{\rho}_n(g_1)_{i,j}\tilde{\rho}_n(g_2 g_3)_{j,k} \\
&= \rho_n(g_1 g_2 g_3)_{i,j}.
\end{aligned}$$

On the other hand, doing a similar computation one can check that

$$\begin{aligned}
&(\Delta \otimes \mathrm{id})(\Delta(\tilde{\rho}_{n,i,j}))(g_1, g_2, g_3) \\
&= \sum_k \tilde{\rho}_{n,i,j}(g_1 g_2)\tilde{\rho}_{n,k,j}(g_3) \\
&= \rho_n(g_1 g_2 g_3)_{i,j}.
\end{aligned}$$

(2)

$$(\mathrm{id} \otimes S) \circ \Delta = (S \otimes \mathrm{id}) \circ \Delta.$$

To show this let $g \in \Gamma$.

$$\begin{aligned}
&(\mathrm{id} \otimes S)(\Delta(\tilde{\rho}_{n,i,j}))(g) \\
&= (\mathrm{id} \otimes S)(\sum_k \tilde{\rho}_{n,i,k} \otimes \tilde{\rho}_{n,k,j})(g) \\
&= \sum_k \rho_{\tilde{n,i,k}}(g)\tilde{\rho}_{n,k,j}(g^{-1}) \\
&= \rho_n(gg^{-1})_{i,j} = \delta_{i,j}.
\end{aligned}$$

On the other hand doing a similar computation

$$\begin{aligned}
&(S \otimes \mathrm{id})(\Delta(\tilde{\rho}_{n,i,j}))(g) \\
&= \rho_n(g^{-1}g) = \delta_{i,j}.
\end{aligned}$$

(3)

$$(\mathrm{id} \otimes \epsilon) \circ \Delta = (\epsilon \otimes \mathrm{id}) \circ \Delta.$$

To show this we proceed as before with $g \in \Gamma$.

$$
\begin{aligned}
&(\mathrm{id} \otimes \epsilon)(\Delta(\tilde{\rho}_{n,i,j}))(g) \\
&= (\mathrm{id} \otimes \epsilon)(\sum_k \tilde{\rho}_{n,i,k} \otimes \tilde{\rho}_{n,k,j})(g) \\
&= \sum_k \rho_{\tilde{n,i,k}}(g)\tilde{\rho}_{n,k,j}(1) \\
&= \rho_n(g.1)_{i,j}.
\end{aligned}
$$

On the other hand doing a similar computation

$$
\begin{aligned}
&(\epsilon \otimes \mathrm{id})(\Delta(\tilde{\rho}_{n,i,j}))(g) \\
&= \rho_n(1.g)_{i,j}.
\end{aligned}
$$

Solution 7: $A \in \mathrm{End}(V)$ for some finite dimensional vector space V. Let $U \subseteq V$ be an A-invariant subspace of V. Let A be semisimple.

(1) We need to show that $A\,|_U$ is semisimple. We can write
$$V = \oplus_{i=1}^r V_{\lambda_i},$$
where V_{λ_i} are the λ_i-eigenspaces and λ_i are the distinct eigenvalues of A. Now if if $u \in U$ such that
$$u = v_1 + \cdots + v_r, \quad v_i \in V_{\lambda_i},$$
then $A^k u = \sum_{i=1}^r \lambda_i^k v_i$. Running $k = 0, \ldots, r-1$ we will have r equations
$$\Lambda \tilde{v} = \tilde{u},$$
where $\tilde{v} = (v_1, \ldots, v_r)^t$, $\tilde{u} = (u, \ldots, A^{r-1}u)^t$ and $\Lambda_{ij} = \lambda_j^{i-1}$ i,e,, a Vandermonde matrix, hence invertible. As $A^i u \in U$ we have $v_i \in U$ for all i. Thus
$$U = \oplus_{i=1}^r V_{\lambda_i} \cap U,$$
hence $A\,|_U$ is semisimple.

(2) We need to show that U has a A-invariant complement. As $A\,|_U$ is semisimple we can choose an eigenabasis E for U. As A is semisimple we can complete E to an eigenbasis F of V. Thus
$$V = U \oplus \mathrm{span}(F \setminus E).$$
$\mathrm{span}(F \setminus E)$ is clearly A invariant being union of eigenspaces of A.

Solution 8: We want to show that a family of commuting semisimple operators is simultaneously diagonalizable. First assume that the family of the commuting semisimple operators is finite, say, $\{A_1, \dots, A_r\}$. We use induction on r. For $r = 1$ it follows by definition. Let $r - 1$ commutating semisimple element are simultaneously diagonalizable. Let E_λ be an eigenspace of A_r. Because of commutativity

$$A_r(A_j v) = A_j(A_r v) = \lambda(A_j v),$$

for $v \in E_\lambda$. That is, E_λ is A_j invariant. Using the previous problem $A_j \mid_{E_\lambda}$ are semisimple. Using induction hypothesis (as $A_j \mid_{E_\lambda}$ also commute) there exists a basis of E_λ such that $A_j \mid_{E_\lambda}$ for $j = 1, \dots, r-1$ are diagonalizable. But all vectors of E_λ are eigenvectors of E_λ. Hence E_λ is also diagonalizable wrt that basis.

Now for possibly infinite family of commutating semisimple operators we consider the span of them inside $\mathrm{End}(V)$. The span would be a finite dimensional vector space as the same is true for $\mathrm{End}(V)$. We choose a basis of the span which would be a finite family of commutating semisimple operators. Apply the previous argument to get a simultaneous eigenbasis for the basis vectors. Hence the claim follows.

Solution 10: Let H be a diagonalizable connected algebraic group. Now the character group $\mathfrak{X}(H)$ of H is a finitely generated abelian group (Lemma 8.3) which thanks to the structure theorem looks like

$$\mathfrak{X}(H) = \mathbb{Z}^r \oplus T,$$

for some $r \geq 0$ and T is the torsion part. On the other hand, $\mathfrak{X}(H)$ is the character group of $\mathbb{G}_m^r \times M$ where M is a group consisting of roots of unity and the character group of M is T. Thus we can conclude that (Corollary 8.12)

$$H \cong \mathbb{G}_m^r \times M.$$

However, H is connected. Hence, the image under the projection map $\pi_T : H \to M$ should be connected. But this is only possible if the discrete group M is trivial. This yields $H \cong \mathbb{G}_m^r$, therefore, a torus.

Solution 11: Let us denote

$$H^{\perp\perp} := \{t \in T : \chi(t) = 1 \text{ for all } \chi \in H^{\perp}\}.$$

Note that, $H^{\perp\perp}$ is a algebraic subgroup of T (as it is defined by polynomial equations) hence a diagonalizable group. We easily see that for all $h \in H$ and for all $\chi \in H^{\perp}$ we have $\chi(h) = 1$, by definition. So we have

$$H \subseteq H^{\perp\perp} \subseteq T.$$

To see the opposite direction we claim that there is a natural surjection of the character groups

$$\mathfrak{X}(T) \twoheadrightarrow \mathfrak{X}(H^{\perp\perp}) \twoheadrightarrow \mathfrak{X}(H)$$

by restriction maps.

We see that if $\chi \in \ker(\mathfrak{X}(T) \twoheadrightarrow \mathfrak{X}(H^{\perp\perp}))$ i.e.

$$\iff \chi\mid_H = 1 \iff \chi \in H^{\perp} \iff \chi\mid_{H^{\perp\perp}} = 1$$

i.e. $\chi \in \ker(\mathfrak{X}(T) \twoheadrightarrow \mathfrak{X}(H))$. Thus $\mathfrak{X}(H) \cong \mathfrak{X}(T)/\ker \cong \mathfrak{X}(H^{\perp\perp})$. Thus $H = H^{\perp\perp}$.

Now we prove the claim. Note that, there is a surjection of the coordinate rings

$$\mathcal{O}(T) \twoheadrightarrow \mathcal{O}(H^{\perp\perp}) \twoheadrightarrow \mathcal{O}(H)$$

again by restriction maps. Thus it is enough to prove that if G is a diagonaligable group with an algebraic subgroup H such that $\mathcal{O}(G) \twoheadrightarrow \mathcal{O}(H)$ then $\mathfrak{X}(G) \twoheadrightarrow \mathfrak{X}(H)$. We know that $\mathfrak{X}(G)$ is a basis of $\mathcal{O}(G)$ (Lemma 8.8) (similarly, for H). Thus

$$I := \{\chi\mid_H : \chi \in \mathfrak{X}(G)\} \subseteq \mathfrak{X}(H),$$

is linear independent. But $\mathfrak{X}(G)$ spans $\mathcal{O}(G)$; which along with the surjection $\mathcal{O}(G) \twoheadrightarrow \mathcal{O}(H)$ imply that I spans $\mathcal{O}(H)$, hence a basis and equals to $\mathfrak{X}(H)$. This concludes the proof.

Another solution: We know that (Theorem 3.11) there exists a finite dimensional representation ρ of G such that $\ker(\rho) = H$. But G is diagonalizable which implies that ρ decomposes as

$$\rho = \oplus_{i=1}^{\dim(\rho)} \chi_i,$$

where $\chi_i\mid_H = 1$. Hence,

$$\ker(\rho) = \cap_{i=1}^{\dim(\rho)} \ker(\chi_i).$$

But

$$H^{\perp\perp} = \cap_{\chi|_H = 1} \ker(\chi) \subseteq \ker(\rho) = H.$$

Thus $H = H^{\perp\perp}$.

Solution 12: We already know that the Zariski closure of a group is an algebraic group. $G(g) \times G(g)$ lies in the closed set

$$\{(a, b) \mid aba^{-1}b^{-1} = 1\},$$

so as $\overline{G(g) \times G(g)}$. If we show that $\overline{G(g) \times G(g)}$ contains $\overline{G(g)} \times \overline{G(g)}$ then we will be done. In other words, we need to show that if a polynomial f vanishes on $X \times Y$ then it will vanish on $\bar{X} \times \bar{Y}$. Now for every $x \in X$ we have $f(x, Y) = 0$. Thus $f(x, \bar{Y}) = 0$, hence f vanishes of $X \times \bar{Y}$. Similarly, for every $y \in \bar{Y}$ we have $f(X, y) = 0$. Hence f vanishes on $\bar{X} \times \bar{Y}$.

As $G(g)$ is commutative we know that (Theorem 10.2) there exists an isomorphism

$$G(g) = G(g)_s \times G(g)_u.$$

Under this map $g \mapsto (g_s, g_u)$ by the main theorem of Jordan decomposition. We know $G(g)_s$ is algebraic, hence it follows that $G(g_s) \subseteq G(g)_s$ and closed. Similarly, $G(g_u) \subseteq G(g)_u$ and closed. Thus $G(g_s) \times G(g_u)$ is closed in $G(g)$. The inverse image of the multiplication map is closed and thus is contained in $G(g_s) \times G(g_u)$. Hence, the multiplication map

$$G(g_s) \times G(g_u) \to G(g),$$

is a surjection. However,

$$G(g_s) \cap G(g_u) = \{1\} = G(g)_s \cap G(g)_u.$$

This forces $G(g)_s = G(g_s)$ and $G(g)_u = G(g_u)$.

Solution 13: We first prove that G_u is normal algebraic subgroup of G. As G is solvable and connected we have that G is trigonalizable (Lie–Kolchin, Theorem 11.2). That is, there exists an embedding $G \hookrightarrow \mathrm{GL}(n)$ such that the elements of G map to upper triangular matrices. Then $G_u = G \cap U_n$ where U_n is the set of upper triangular unipotent matrices. We consider the natural morphism

$$G \hookrightarrow \mathrm{GL}(n) \to D_n,$$

where D_n is the set of diagonal matrices. Kernel of this map is $G \cap U_n = G_u$. Hence G_u is a normal algebraic subgroup of G.

Now we show that G_s is normal algebraic subgroup. We know that $G_s \subset Z(G)$ (Lemma 12.2). We will prove that G_s is an algebraic subgroup. The elements of G_s are semisimple and commuting. If $G \subset \mathrm{GL}(V)$ we can have common eigenspaces V_λ for all of G_s. As $G_s \subset Z(G)$ we have that G will leave each eigenspace invariant. We apply Lie–Kolchin to each V_λ to get a closed embedding of G into upper triangular matrices in $\mathrm{GL}(V)$ so that G_s gets mapped to set of diagonal matrices. Hence, G_s is a central, therefor normal, algebraic subgroup.

To prove $G_s \times G_u \to G$ an isomorphism we first see that the map is injective as because of the above embeddings $G_s \cap G_u = \{1\}$. Surjectivity follows from the Jordan decomposition. The map is clearly a morphism. To see that the inverse map is also a morphism we note that the map $g \mapsto g_s$ through the Jordan decomposition is a morphism , hence so is $g \mapsto g_s^{-1}g$. This concludes the proof.

Solution 14:

(a) See Proposition 11.3 in Szamuely's notes.

(b) We know that k^{n^2} is irreducible, since its coordinate ring is an integral domain. The group $G := \det^{-1}(k \setminus \{0\})$ is non-empty and Zariski open, hence Zariski dense in k^{n^2}. As a set is irreducible if and only if its Zariski closure is, we see that G is irreducible. It remains to show that G is homeomorphic to

$$\mathrm{GL}_n(k) = \{(x,t) \in k^{n^2+1} \mid \det(x)t = 1\}.$$

Clearly the projection from $\mathrm{GL}_n(k)$ to G is bijective and continuous. Let us show it is also open. A basic open set U in $\mathrm{GL}_n(k)$ is of the form

$$U = \left\{(x,t) \in \mathrm{GL}_n(k) \mid \sum_{i=0}^{d} p_i(x)t^i \neq 0\right\}$$

for some $d \in \mathbb{N}$ and polynomials p_i in x. The projection of U to G is then given by

$$\left\{x \in G \mid \sum_{i=0}^{d} p_i(x)\det(x)^{-i} \neq 0\right\} = \left\{x \in G \mid \sum_{i=0}^{d} p_i(x)\det(x)^{d-i} \neq 0\right\},$$

where we cleared the denominators by multiplying with the non-zero number $\det(x)^d$. The latter description exhibits the projection of U as open inside G.

(c) By linear algebra, $\mathrm{GL}_n(k)$ acts transitively on bases of k^n. Given m-dimensional subspaces V, W of k^n, choose bases $(v_i)_{i=1}^m, (w_i)_{i=1}^m$ of V, W, respectively, and enlarge them to bases $(v_i)_{i=1}^n, (w_i)_{i=1}^n$ of k^n. Then some $g \in \mathrm{GL}_n(k)$ sends $(v_i)_{i=1}^n$ to $(w_i)_{i=1}^n$, hence V to W.

(d) Note first that the action of $\mathrm{GL}_n(k)$ on wedge products defines an algebraic representation $\mathrm{GL}_n(k) \to \mathrm{GL}(\bigwedge^m k^n) \cong \mathrm{GL}_N(k)$, where $N = \binom{n}{m}$. Indeed, one can check that the entries of the matrix representation of $g \in \mathrm{GL}_n(k)$ acting on $\bigwedge^m k^n$ with respect to the basis $(e_{i_1} \wedge \ldots \wedge e_{i_m})_{1 \le i_1 < \cdots < i_m \le n}$ are the $m \times m$-minors of g. Thus, the claim that the action of $\mathrm{GL}_n(k)$ on $\mathbb{P}^N$ induced by the exterior power representation is algebraic follows from the more general claim that the action of $\mathrm{GL}_N(k)$ on $\mathbb{P}^N$ (induced by the linear action of $\mathrm{GL}_N(k)$ on k^N) is algebraic, which is clear since the action map

$$\begin{aligned} \mathrm{GL}_N(k) \times \mathbb{P}^N &\to \mathbb{P}^N, \\ (g, [x]) &\mapsto [gx], \end{aligned}$$

has as components homogeneous (indeed, linear) polynomials in $x_1, \ldots, x_N$ (cf. the argument in Example 12.4(2) in Szamuely's notes).

Finally, irreducibility of $G_{m,n}$ follows by combining all of the above, since, by transitivity, $G_{m,n}$ is the image of the irreducible set $\mathrm{GL}_n(k)$ under an orbit map, and continuous images of irreducible sets are irreducible.