INTRODUCTION TO ALGEBRAIC GROUPS
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SYNOPSIS OF LECTURES

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Contents

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1. Introduction

Let k be an algebraically closed field. Let $k[x] := k[x_1, \dots, x_n]$ denote the polynomial ring in n variables.

Definition 1.1. For a subset S of $k[x]$, the *vanishing locus* is the subset $V(S)$ of k^n defined by

$$V(S) := \{p \in k^n : f(p) = 0 \text{ for all } p \in S\}. \tag{1.1}$$

A subset X of k^n is *algebraic* if it is of the form $X = V(S)$ for some S.

In words, an algebraic set is one that can be defined via polynomial equations.

Example 1.2. The algebraic subsets X of $k = k^1$ are either finite or equal to k. Explicitly,

$$k = V(\emptyset), \quad \{p_1, \dots, p_r\} = V(\{\prod_{j=1}^{r}(x - p_j)\}). \tag{1.2}$$

Example 1.3. The multiplicative group $k^\times = k - \{0\}$ is *not* an algebraic subset of k, but may be identified with an algebraic subset of k^2 via the map

$$\begin{gathered} k^\times \xrightarrow{\cong} \{(x, y) \in k^2 : xy = 1\} = V(xy - 1) \\ x \mapsto (x, 1/x). \end{gathered} \tag{1.3}$$

Example 1.4. Let $M_n(k) \cong k^{n^2}$ denote the space of $n \times n$ matrices. The group $\mathrm{GL}_n(k)$ is *not* an algebraic subset of $M_n(k)$, but may be identified with an algebraic subset of $M_n(k)^2$ via the map

$$\begin{gathered} \mathrm{GL}_n(k) \xrightarrow{\cong} \{(x, y) \in M_n(k)^2 : xy = 1\} = V(\{\sum_k x_{ik}y_{kj} - \delta_{ij} : i, j = 1, \dots, n\}) \\ x \mapsto (x, x^{-1}). \end{gathered} \tag{1.4}$$

Definition 1.5. A *linear algebraic group* over k is a subgroup G of some general linear group $\mathrm{GL}_n(k)$ that is an algebraic subset of $M_n(k)^2$ with respect to the above embedding.

In words, a linear algebraic group is a subgroup of $\mathrm{GL}_n(k)$ that can be defined using polynomial equations in the entries of a matrix and its inverse.

Example 1.6. The groups $k^\times = \mathrm{GL}_1(k), \mathrm{GL}_n(k), \mathrm{SL}_n(k) = \{x \in \mathrm{GL}_n(k) : \det(x) = 1\}, \mathrm{O}_n(k) = \{x \in \mathrm{GL}_n(k) : x^t x = 1\}$, $\mathrm{SO}_n(k) = \{x \in \mathrm{SL}_n(k) : x^t x = 1\}$ are all evidently linear algebraic groups, as are the groups

$$\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \cap \mathrm{GL}_3(k), \qquad \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \cap \mathrm{GL}_3(k).$$

We sometimes write $\mathbb{G}_m(k) := k^\times$ ("multiplicative group"). The group $\mu_n(k) := \{z \in k^\times : z^n = 1\}$ of nth roots of unity in k is a linear algebraic group. The additive group $\mathbb{G}_a(k) := k$

may be identified with the linear algebraic group $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \cap \mathrm{SL}_2(k)$ via the map $x \mapsto \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$. To give a non-example, for $k = \mathbb{C}$, the unitary groups $\mathrm{U}(n)$ are *not* linear algebraic groups.

The plan for the course is to cover roughly the following topics:

- Basics on algebraic groups, Hopf algebras, etc.
- Jordan decomposition
- Useful examples of actions (flag varieties, toric varieties, ...), quotients
- Structure theory: Borel subgroups, maximal tori, reductive groups, roots, classification

2. Algebraic groups and Hopf algebras

Definition 2.1. Let $X \subseteq k^m$ and $Y \subseteq k^n$ be algebraic sets. A *morphism* (or *regular map*) $f : X \to Y$ is a map described in coordinates by polynomials, thus $f(p) = (f_1(p), \dots, f_n(p))$ for some $f_1, \dots, f_n \in k[x_1, \dots, x_m]$. We denote by $\mathrm{Hom}(X, Y)$ the set of such morphisms.

This definition equips the class of algebraic sets with the structure of a category; in particular, we may speak of isomorphisms.

Lemma 2.2. *Let $X \subseteq k^m$ and $Y \subseteq k^n$ be algebraic sets. Then the product set $X \times Y \subseteq k^{m+n} \cong k^m \times k^n$ is algebraic.*

Definition 2.3. An *algebraic group* is an algebraic set G equipped with an element $e \in G$ and a pair of morphisms $m : G \times G \to G$ and $i : G \to G$ satisfying the group axioms (with e, m, i playing the roles of identity element, multiplication and inversion).

A *morphism* of algebraic groups $f : G \to H$ is a group homomorphism that is also a morphism of the underlying algebraic sets.

This definition equips the class of algebraic groups with the structure of a category.

Example 2.4. Every linear algebraic group is an algebraic group in the above sense. Conversely, we will see below that every algebraic group is isomorphic to a linear algebraic group.

[1]

Definition 2.5. Let X be a subset of k^n. The *vanishing ideal* of X is the ideal $I(X)$ of $k[x_1, \dots, x_n]$ defined by

$$I(X) := \{f : f(p) = 0 \text{ for all } p \in X\}. \tag{2.1}$$

Definition 2.6. Let R be a ring (commutative, unital) and $I \subseteq R$ an ideal. The *radical* of I is the ideal

$$\sqrt{I} := \{f \in R : f^m \in I \text{ for some } m\}. \tag{2.2}$$

[1]End of lecture #1, 17 Feb 2020

The ideal I is called *radical* if $\sqrt{I} = I$, i.e., if $f^m \in I \implies f \in I$. The ring R is called *reduced* if the zero ideal is radical, i.e., if $f^m = 0 \implies f = 0$.

Theorem 2.7 (Hilbert's Nullstellensatz)**.** *For any ideal $I \subseteq k[x_1, \ldots, x_n]$, we have*

$$I(V(I)) = \sqrt{I}. \tag{2.3}$$

Consequently the maps $I \mapsto V(I)$ and $X \mapsto I(X)$ define mutually-inverse inclusion-reversing bijections

$$\{\text{algebraic subsets } X \subseteq k^n\} \leftrightarrow \{\text{radical ideals } I \subseteq k[x_1, \ldots, x_n]\}. \tag{2.4}$$

Definition 2.8. Let $X \subseteq k^n$ be an algebraic set. The *coordinate ring* of X is the k-algebra $k[X]$ defined by

$$k[X] := k[x]/I(X) := k[x_1, \ldots, x_n]/I(X). \tag{2.5}$$

Remark 2.9. To each $f \in k[X]$ we may associate a function $f : X \to k$. This association induces an identification

$$k[X] = \operatorname{Hom}(X, k). \tag{2.6}$$

Elements of the coordinate ring, when identified with functions in this way, are called *regular functions.*

Remark 2.10. It's clear that any such coordinate ring $k[X]$ is a finitely-generated reduced k-algebra.

Lemma 2.11. *Let X, Y be algebraic sets. The map*

$$k[X] \otimes k[Y] \to k[X \times Y] \tag{2.7}$$

$$f \otimes g \mapsto [(p, q) \mapsto f(p)g(q)] \tag{2.8}$$

is an isomorphism.

Theorem 2.12. *Let X, Y be algebraic sets. Write* $\operatorname{Hom}(X, Y)$ *for morphisms of algebraic sets and* $\operatorname{Hom}(k[Y], k[X])$ *for homomorphisms of k-algebras. The map*

$$\operatorname{Hom}(X, Y) \to \operatorname{Hom}(k[Y], k[X]) \tag{2.9}$$

$$\phi \mapsto \phi^* : f \mapsto f \circ \phi \tag{2.10}$$

is a bijection, with inverse described by

$$\phi = (\phi^*(\bar{y}_1), \ldots, \phi^*(\bar{y}_n)). \tag{2.11}$$

Theorem 2.13. *Every finitely-generated reduced k-algebra is isomorphic to $k[X]$ for some algebraic set X.*

Corollary 2.14. *The contravariant functor $(X \mapsto k[X], \phi \mapsto \phi^*)$ defines an anti-equivalence of categories*

$$\{\text{algebraic sets over } k\} \leftrightarrow \{\text{finitely-generated reduced } k\text{-algebras}\}. \tag{2.12}$$

Corollary 2.15. *A pair of algebraic sets X and Y are isomorphic if and only if their coordinate rings $k[X]$ and $k[Y]$ are isomorphic as k-algebras.*

Lemma 2.16. *Let X, Y be algebraic sets. The following are equivalent:*
(i) X is isomorphic to some algebraic subset of Y.
(ii) There is a surjective homomorphism $k[Y] \twoheadrightarrow k[X]$.

Example 2.17. Write $*$ for an algebraic set consisting of a single point. We have $k[*] = k$. For any algebraic set X, we may identify $X \cong \operatorname{Hom}(*, X) \cong \operatorname{Hom}(k[X], k[*]) = \operatorname{Hom}(k[X], k)$.

Let G be an algebraic group. Write $A := k[G]$ for its coordinate ring. The data (m, i, e) equips A with the additional structure $(\Delta, \iota, \varepsilon)$, where $\Delta = m^*$, $\iota = i^*$, and $\varepsilon = e^*$, where we identify e with a map $* \to G$ as above. Thus

$$\Delta : A \to A \otimes A,$$
$$\iota : A \to A,$$
$$\varepsilon : A \to k.$$

Example 2.18. For $G = \mathbb{G}_a(k)$, we have $k[G] \cong k[x]$. and $\Delta(x) = x \otimes 1 + 1 \otimes x$. For $G = \mathbb{G}_m(k)$, we have $k[G] \cong k[x, y]/(xy - 1) \cong k[x, 1/x]$. and $\Delta(x) = x \otimes x$. For $G = \mathrm{GL}_n(k)$, we have $k[G] \cong k[\{x_{ij}\}, 1/\det]$ (to be explained later) and $\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$.

The group axioms on (m, i, e) may be reformulated in terms of $(\Delta, \iota, \varepsilon)$ (see page 9 of Szamuely's notes).

Definition 2.19. A *Hopf algebra* A over k is an k-algebra equipped with a tuple $(\Delta, \iota, \varepsilon)$ satisfying the reformulated group axioms. A *morphism* of Hopf algebras $A \to B$ is an algebra morphism that commutes with the additional structure.

Theorem 2.20. *The contravariant functor $G \mapsto k[G], \phi \mapsto \phi^*$ defines an anti-equivalence of categories*

$$\{\textit{algebraic groups over } k\} \leftrightarrow \{\textit{finitely-generated Hopf algebras over } k\} \tag{2.13}$$

[2]

2.1. **An aside on GL_n.**

Lemma 2.21. *Set*

$$G_1 := \{(x, y) \in M_n(k)^2 : xy = 1\},$$
$$G_2 := \{(x, t) \in M_n(k) \times k : \det(x)t = 1\}.$$

Then the maps

$$G_1 \to G_2$$
$$(x, y) \mapsto (x, \det(y))$$

[2] End of lecture #2, 19 Feb 2020

and

$$G_2 \to G_1$$
$$(x,t) \mapsto (x,y)$$

where $y_{ij} := (-1)^{i+j} t X_{ij}$*, with* X_{ij} *the* (j,i) *minor of* x*, are mutually inverse isomorphisms of algebraic groups.*

Thus the two algebraic group structures on $\mathrm{GL}_n(k)$ defined earlier coincide.

Lemma 2.22. $k[\mathrm{GL}_n(k)] = k[x_{11}, \dots, x_{nn}, 1/\det]$.

Lemma 2.23. *Let* V *be a finite-dimensional vector space (over* k*). Choosing a basis of* V *defines a group isomorphism* $\mathrm{GL}(V) \cong \mathrm{GL}_n(k)$*, hence equips* $\mathrm{GL}(V)$ *with the structure of an algebraic group. This structure is independent of the choice of basis.*

3. Actions and representations

Definition 3.1. Let G be an algebraic group. By a *right* G*-set* we mean algebraic set M equipped with a morphism

$$\alpha : M \times G \to M$$
$$(m,g) \mapsto \alpha(m,g) =: mg,$$

called the *action map*, satisfying the familiar axioms

$$me = m, \quad (mg_1)g_2 = m(g_1 g_2). \tag{3.1}$$

We may similarly define algebraic left G-sets.

Example 3.2. Any algebraic group G acts on itself by right translation. GL_n acts on k^n, regarded as a space of row vectors, by matrix multiplication on the right.

Remark 3.3. A right G-set M may be described in terms of a morphism $k[M] \to k[M] \otimes k[G]$ satisfying axioms dual to (3.1). Those axioms equip the space $k[M]$ with the structure of what is called a *comodule* for the Hopf algebra $k[G]$.

Definition 3.4. Let G be an algebraic group and V a vector space (over the same field k, possibly infinite-dimensional). By a *representation* of G on V we will mean a group homomorphism $r_V : G \to \mathrm{GL}(V)$ such that there is a linear map $\Delta : V \to V \otimes k[G]$ with the following property. Let $v \in V$. Write $\Delta v = \sum_i v_i \otimes f_i$. Then for all $g \in G$,

$$r_V(g)v = \sum_i f_i(g) v_i. \tag{3.2}$$

(Equivalently, V is a comodule for $k[G]$.)

Given two representations V and W, a linear map $\phi : V \to W$ is called *equivariant*, or a *morphism of representations*, if we have

$$\phi(r_V(g)v) = r_W(g)\phi(v) \tag{3.3}$$

for all $v \in V$, $g \in G$.

Example 3.5. Let M be a right G-set. Then G acts by automorphisms on M, hence also on $k[M]$, giving a homomorphism

$$\rho : G \to \mathrm{GL}(k[M])$$
$$\rho(g)v(m) = v(mg).$$

The action map $M \times G \to M$ gives rise to an algebra morphism $\Delta : k[M] \to k[M] \otimes k[G]$ characterized by $\Delta v(m, g) = v(mg)$. If $\Delta v = \sum_i v_i \otimes f_i$, then

$$\rho(g)v(m) = v(mg) = \Delta v(m, g) = \sum_i f_i(g)v_i(m). \tag{3.4}$$

Thus ρ is a representation of G. When M is G with the action given by right translation, the resulting $\rho : G \to \mathrm{GL}(k[G])$ is called the *right regular representation* of G.

Lemma 3.6. *Let G be an algebraic group. Let V be a finite-dimensional vector space. Let $\rho : G \to \mathrm{GL}(V)$ be a group homomorphism. The following are equivalent:*

(1) ρ is a morphism of algebraic groups.
(2) ρ is a representation.

Definition 3.7. Let G be an algebraic group and V a representation of G. A subspace W of V is called *invariant* or said to be a *subrepresentation* if $\rho(g)W \subseteq W$ for all $g \in G$.

Lemma 3.8. *For a subspace W of a representation V, the following are equivalent:*

(1) V is invariant.
(2) $\Delta(W) \subseteq W \otimes k[G]$.

In particular, a subrepresentation defines a representation.

Lemma 3.9. *Any representation V is a filtered union of its finite-dimensional subrepresentations, that is to say:*

(1) Each $v \in V$ belongs to some finite-dimensional subrepresentation W of V.
(2) For any two finite-dimensional subrepresentations W_1, W_2 of V are contained in some common finite-dimensional subrepreentation W of V.

Theorem 3.10. *Every algebraic group G is isomorphic to a linear algebraic group.*

Proof. Let $\rho : G \to \mathrm{GL}(k[G])$ be the right regular representation. By the Hilbert basis theorem, $k[G]$ admits a finite set of generators. By Lemma 3.9, we may find a finite-dimensional subrepresentation V of $k[G]$ that generates $k[G]$. By Lemma 3.6, the induced map $\phi : G \to \mathrm{GL}(V)$ is a morphism of algebraic groups. Consider the dual map

$$\phi^* : k[\mathrm{GL}(V)] \to k[G].$$

We claim that ϕ^* is surjective. Since ϕ^* is an algebra morphism, it is ejough to check that $V \subseteq \mathrm{image}(\phi^*)$. To see this, let $\ell : V \to k$ be the linear functional given by evaluation at the identity element, thus $\ell(f) = f(e)$. Let $f \in V$. Then $a(g) := \ell(gf)$ defines a regular function on $\mathrm{GL}(V)$; indeed, it is given by a linear combination of the matrix entries of g with respect to any basis. We compute that

$$\phi^*(a)(g) = a(\rho(g)) = \ell(\rho(g)f) = f(eg) = f(g),$$

from which it follows that

$$f = \phi^*(a) \in \mathrm{image}(\phi^*).$$

Since f was arbitrary, we conclude as required that ϕ^* is surjective.

Since V generates $k[G]$, it follows that ϕ^* is surjective. Thus, by (the proof of) Lemma 2.16, ϕ defines an isomorphism of algebraic sets between G and some algebraic subset H of $\mathrm{GL}(V)$. Since ϕ is a group homomorphism, we deduce that H is an algebraic subgroup of $\mathrm{GL}(V)$ and that ϕ is an isomorphism of algebraic groups. $\square$

[3]

Theorem 3.11. *Let H be an algebraic subgroup of an algebraic group G. There is a finite-dimensional representation V of G and a line $L \subseteq V$ so that H is the stabilizer in G of L.*

Theorem 3.12. *Let H be a normal algebraic subgroup of an algebraic group G. Then there is a finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ such that $H = \ker(\rho)$.*

Proof. Start with a finite-dimensional representation $\phi : G \to \mathrm{GL}(V)$ and a nonzero vector $v_0 \in V$ for which H is the stabilizer of the line $\langle v_0 \rangle$ spanned by G. In particular, v_0 is a common eigenvector of all $h \in H$. Let $E \subseteq V$ denote the set of common eigenvectors of all $h \in H$. Let V_H denote the span of E.

We claim that for all $v \in E$ and $g \in G$, we have $gv \in E$. To see this, let $h \in H$. Write $hgv = g(g^{-1}hg)v$. Since H is normal, we have $g^{-1}hg \in H$. Thus $(g^{-1}hg)v$ is a multiple of v, hence hgv is a multiple of gv, i.e., gv is an eigenvector of h. Since h was arbitrary, we deduce that $gv \in E$, as required.

It follows that V_H is an invariant subspace. By shrinking V if necessary, we may and shall assume that $V_H = V$. Consider the decomposition

$$E = \sqcup_\chi E_\chi,$$

where χ runs over functions $\chi : H \to k^\times$ and $E_\chi = \{v \in E : hv = \chi(h)v \text{ for all } h \in H\}$. Consider the associated decomposition $V = \oplus_\chi V_\chi$ of V into distinct common eigenspaces of all $h \in H$, where $V_\chi = \langle E_\chi \rangle = E_\chi \cup \{0\}$. Let $W := \oplus_\chi \mathrm{End}(V_\chi) \leq \mathrm{End}(V)$ denote the space of endomorphisms leaving each V_χ invariant. We define a representation $\tilde{\gamma}$ of G on $\mathrm{End}(V)$ by conjugation via ϕ: for $g \in G$ and $\lambda \in \mathrm{End}(V)$,

$$\tilde{\gamma}(g)\lambda := \phi(g) \circ \lambda \circ \phi(g)^{-1}.$$

This action stabilizes W, because the normality of H implies that G permutes the common eigenspaces of H. We obtain a subrepresentation $\gamma : G \to \mathrm{GL}(W)$.

[3]End of lecture #3, 24 Feb 2020

We claim that $H = \ker(\gamma)$. To see this, let $h \in H$ and $\lambda \in W$. Since λ stabilizes each V_χ and $\phi(h)$ and its inverse act on each V_χ by scalars, we see that $h \cdot \lambda = \phi(h) \circ \lambda \circ \phi(h)^{-1} = \lambda$. Thus $H \subseteq \ker(\rho)$. Conversely, let $g \in \ker(\rho)$. Since W contains the projection onto V_χ, we see that $\phi(g)$ acts on each V_χ. Also, the restriction of $\phi(g)$ to V_χ commutes with $\mathrm{End}(V_\chi)$, hence is a scalar operator. In particular, $\phi(g)$ stabilizes the line $\langle v_0 \rangle$ and thus g belongs to H, as required. □

4. Connected components

Lemma 4.1. *Let I_1, I_2 and I_λ ($\lambda \in \Lambda$) be ideals of $k[x_1, \ldots, x_n]$.*

- *We have $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$.*
- *We have $V(\cup_\lambda I_\lambda) = \cap_\lambda V(I_\lambda)$.*

Definition 4.2. The *Zariski topology* on k^n is that for which the closed sets are the algebraic sets. More generally, we equip any algebraic set with the induced topology.

Lemma 4.3. *For any algebraic set X, the open subsets of the form $D(f) := \{p \in X : f(p) \neq 0\}$, for $f \in k[X]$, form an open basis for the Zariski topology.*

Definition 4.4. We say that a topological space X is *irreducible* if any of the following equivalent conditions are satisfied:

- X cannot be written as the union $Z_1 \cup Z_2$ of two closed proper subsets Z_1, Z_2.
- Any two nonempty open subsets U_1, U_2 of X have nonempty intersection.
- Any nonempty open subset of X is dense.

By comparison, recall that X is *connected* if it satisfies the weaker condition that it cannot be written as the *disjoint* union or two closed proper subsets, or equivalently, if it cannot be *covered* by a pair of nonempty open subsets. Thus

$$\text{irreducible} \implies \text{connected}.$$

The converse fails: one can check that the union of coordinate axes $V(xy) \subseteq k^2$ is connected, but not irreducible.
[4]

Lemma 4.5. *The following are equivalent for an algebraic set $X \subseteq k^n$:*

(i) X is irreducible.
(ii) $k[X]$ is a domain.
(iii) $I(X) \subseteq k[x_1, \ldots, x_n]$ is a prime ideal.

Proof. X is irreducible iff every pair of nonempty open subsets U_1, U_2 have nonempty intersection iff every pair of nonempty *basic* open subsets $D(f_1), D(f_2)$ have nonempty intersection. By the Nullstellensatz, the latter condition is equivalent to asking that $f_1 f_2$ be nonzero whenever f_1 and f_2 are nonzero. □

[4]End of lecture #4, 26 Feb 2020

More generally, we obtain bijections

$$\{\text{closed subsets of } X\} \leftrightarrow \{\text{radical ideals of } k[X]\},$$

$$\{\text{irreducible closed subsets of } X\} \leftrightarrow \{\text{prime ideals of } k[X]\}$$

$$X = \{\text{points of } X\} \leftrightarrow \{\text{maximal ideals of } k[X]\}$$

given by sending $Y \subseteq X$ to $I_X(Y) := \{f \in k[X] : f|_Y = 0\}$.

Lemma 4.6. *An algebraic set X is connected if and only if the only idempotents e in $k[X]$ (i.e., elements satisfying $e^2 = e$) are 0 and 1, or equivalently, if there is no nontrivial way to write $k[X]$ as a product of two k-algebras.*

Lemma 4.7. *Any radical ideal I of $k[x_1, \ldots, x_n]$ may be written uniquely as the intersection $I = \cap P_i$ of finitely many minimal prime ideals $P_i \supseteq I$. Each minimal prime P_i does not contain $\cap_{j:j\neq i} P_j$.*

Definition 4.8. Let X be an algebraic set. By a *component* of X we mean a maximal closed irreducible subset, i.e., a closed irreducible subset Z of X that is not strictly contained in any larger closed irreducible subsets.

Lemma 4.9. *Any algebraic set X has finitely many components Z_i, We have the decomposition. $X = \cup_i X_i$. Each component Z_i is not contained in the union $\cup_{j:j\neq i} X_j$ of the other components.*

Example 4.10. The decomposition of $X = V(xy) \subseteq k^2$ into components is $X = Z_1 \cup Z_2$, where $Z_1 = V(x), Z_2 = V(y)$.

Example 4.11. Let X be a *finite* algebraic set. Then every subset of X is closed, hence X carries the discrete topology. Moreover, one can check using the Chinese remainder theorem that for every algebraic set Y, every function $f : X \to Y$ is a morphism. In particular, every function $f : X \to k$ is regular. It follows that the category of finite algebraic sets is equivalent to the category of finite sets. If $X = \{p_1, \ldots, p_r\}$, then the components of X are the $\{p_j\}$.

Lemma 4.12. *Morphisms of algebraic sets are continuous. Isomorphisms are homeomorphisms. In particular, they map irreducible components to irreducible components. In particular, an automorphism of an algebraic set permutes the components.*

Theorem 4.13. *Let G be an algebraic group. There is a unique irreducible component G^0 of G containing the identity element e. The irreducible components of G are precisely the cosets of G^0. There are finitely many such components. They are also the connected components. In particular, G is irreducible iff it is connected.*

Remark 4.14. We'll see on the homework that if $k = \mathbb{C}$, then G is connected in the Zariski sense iff G is connected in the "usual" sense as a manifold. In particular, the manifold underlying a complex algebraic group has only finitely many connected components.

Example 4.15. It turns out that the groups $G = \mathrm{GL}_n(k), \mathrm{SL}_n(k), \mathrm{SO}_n(k), \mathrm{Sp}_{2n}(k)$ are connected, i.e., satisfy $G^0 = G$. On the other hand, if $G = \mathrm{O}_n(k)$, then $G^0 = \mathrm{SO}_n(k)$, and $G/G_0 \cong \mu_2(k) = \{\pm 1\}$.

Example 4.16. The category of finite algebraic groups is equivalent to the category of finite groups. For a finite (algebraic) group G we have $G^0 = \{e\}$ and $G/G^0 = G$.

5. Jordan decomposition

Definition 5.1. Let V be a finite-dimensional vector space (always over k, an algebraically closed field). Let $g \in \mathrm{End}(V)$. We say that g is *semisimple* if it admits a basis of eigenvectors; equivalently, the minimal polynomial of g has distinct roots. We say that g is *nilpotent* if some power g^n is zero. We say that g is *unipotent* if $g - 1$ is nilpotent.

Lemma 5.2. *If $x \in \mathrm{End}(V)$ is semisimple and nilpotent, then $x = 0$. If $g \in \mathrm{GL}(V)$ is semisimple and unipotent, then $g = 1$.*

Lemma 5.3. *The product of two commuting semisimple (resp. unipotent) elements is semisimple (resp. unipotent).*

Our goal for this section is to prove the following:

Theorem 5.4. *Let G be an algebraic group. For each $g \in G$ there are unique elements $g_s, g_u \in G$ with the following properties:*

(i) $g = g_s g_u = g_u g_s$
(ii) For each representation V of G, $r_V(g_s)$ is semisimple and $r_V(g_u)$ is unipotent.

[5]

Lemma 5.5. *Let V be a finite-dimensional vector space and $g \in \mathrm{GL}(V)$. There are unique $g_s, g_u \in \mathrm{GL}(V)$, with g_s semisimple and g_u unipotent, so that $g = g_s g_u = g_u g_s$. Moreover, g_s and g_u belong to the subalgebra $k[g]$ of $\mathrm{End}(V)$ generated by g.*

Proof. For the existence, we appeal to the Jordan decomposition. For instance,

$$g = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \implies g_s = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}, \quad g_u = \begin{pmatrix} 1 & \lambda^{-1} & 0 \\ & 1 & \lambda^{-1} \\ & & 1 \end{pmatrix}. \tag{5.1}$$

More intrinsically, consider the characteristic polynomial $\det(T - g) = \prod_i (T - \lambda_i)^{n_i}$ and the decomposition $V = \oplus_i V_i$ into generalized eigenspaces $V_i = \ker((g - \lambda_i)^{n_i})$. Then g_s acts on V_i by the scalar λ_i, whereas $g_u := g g_s^{-1}$.

Next, using the Chinese remainder theorem, verify that $g_s, g_u \in k[g]$. For instance, choose $Q \in k[T]$ so that $Q \equiv \lambda_i \pmod{(T - \lambda_i)^{n_i}}$ for all i (it is possible to do this because the factors $T - \lambda_i$ are pairwise relatively prime). Then $Q(g) = g_s$. We may similarly find a polynomial Q with $Q(g) = g_s^{-1}$, in which case $gQ(g) = g_u$.

For the uniqueness, let $g = su = us$ be any other such decomposition. Note that s and u commute with g, hence with anything in $k[g]$, hence with g_s and g_u. Then use that a product of commuting semisimple (resp. unipotent) elements has the same property. Then use that only the trivial element is both semisimple and unipotent. □

Lemma 5.6. *Any g-invariant subspace W of V is invariant by g_s and g_u.*

[5] End of lecture #5, 2 Mar 2020

Lemma 5.7. *For any linear map $\phi : V \to W$, $\alpha \in \mathrm{GL}(V)$ and $\beta \in \mathrm{GL}(W)$ such that $\phi \circ \alpha = \beta \circ \phi$, we have $\phi \circ \alpha_s = \beta_s \circ \phi$ and $\phi \circ \alpha_u = \beta_u \circ \phi$.*

Lemma 5.8. *For $\alpha, \beta \in \mathrm{GL}(V)$, we have $(\alpha \otimes \beta)_s = \alpha_s \otimes \beta_s$ and $(\alpha \otimes \beta)_u = \alpha_u \otimes \beta_u$.*

Proof of Theorem 5.4. Consider the families of operators $\lambda_V \in \mathrm{End}(V)$ defined by $\lambda_V := r_V(g)_s$, or the similar family defined by $\lambda_V := r_V(g)_u$. Observe that this family has the following properties:

- (i) Denote by k the one-dimensional representation equipped with the trivial action $r(g) = 1$. Then $\lambda_k = 1$ (the identity operator).
- (ii) $\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$.
- (iii) For all G-equivariant maps $\phi : V \to W$, we have $\lambda_W \circ \phi = \phi \circ \lambda_V$.

We may conclude by the following lemma. □

Lemma 5.9. *Let G be an algebraic group. Suppose given, for each finite-dimensional representation V of G, an endomorphism $\lambda_V \in \mathrm{End}(V)$ satisfying the above properties. Then there exists a unique $g \in G$ so that $\lambda_V = r_V(g)$ for all V.*

[6]

Proof. (Sketch; details given in lecture.)

We verify first that we may extend the assignment $V \mapsto \lambda_V$ to infinite-dimensional representations, by glueing. Then we consider the right regular representation $A := k[G]$. We obtain $\lambda_A : A \to A$. The multiplication map $A \otimes A \to A$ is equivariant for the representations $r_A \otimes r_A$ and r_A, from which it follows that $\lambda_A(fg) = \lambda_A(f)\lambda_A(g)$, i.e., that λ_A is an algebra morphism, hence corresponds to a morphism of algebraic sets $\phi : G \to G$. Also, r_A commutes with the left regular representation, from which it follows that ϕ commutes with left translation. Thus, setting $g := \phi(e)$, we have $\phi(h) = \phi(he) = h\phi(e) = hg$. Thus $\phi^*(f)(x) = f(xg)$, i.e., $\lambda_A = \phi^* = r_A(g)$.

It remains to show that for every V, we have $\lambda_V = r_V(g)$. To that end, we may assume for convenience that V is finite-dimensional. Let V_0 denote the same vector space as V, but with the trivial action of G. The map $\Delta : V \to V \otimes A$ then defines an equivariant morphism of representations

$$V \to V_0 \otimes A.$$

This map is injective, since its composition with $v \otimes f \mapsto f(1)v$ is the identity map $V \to V$. We now deduce from the hypotheses that $\lambda_V = r_V(g)$. (Check first, using (i) and (iii), that λ_{V_0} is the identity map. Thus $\lambda_{V_0 \otimes A}$ and also λ_V are determined by $\lambda_A = r_A(g)$.) □

Corollary 5.10. *Let $f : G \to H$ be a morphism of algebraic groups. Then for each $g \in G$, we have $f(g_s) = f(g)_s$ and $f(g_u) = f(g)_u$.*

[6] End of lecture #6, 9 Mar 2020

Proof. Take any faithful (i.e., injective) representation $\rho : H \hookrightarrow \mathrm{GL}(V)$, which exists by Theorem 3.10. Then $\rho \circ f$ is a representation of G, so by the above theorem applied to both $\rho \circ f$ and ρ, we obtain

$$\rho(f(g_s)) = \rho(f(g))_s = \rho(f(g)_s), \quad \rho(f(g_u)) = \rho(f(g))_u = \rho(f(g)_u).$$

Since ρ is injective, the required conclusion follows. □

Definition 5.11. Let G be an algebraic group. We say that $g \in G$ is *semisimple* (resp. *unipotent*) if $g = g_s$ (resp. if $g = g_u$).

6. An aside on non-commutative algebra

In the next section we'll need a consequence of the Jacobson density theorem. The purpose of this section is to state that consequence and record its proof.

In the final 25 or so minutes of the lecture on Mar 11, we gave a crash course on non-commutative algebra: non-commutative rings, left/right/two-sided ideals, quotients, left/right modules, Schur's lemma, Jacobson density theorem. As mentioned, most of these concepts are not needed in the rest of the course. Moreover, they are not really needed to prove the consequences of the Jacobson density theorem that we'll need. We'll state those consequences below and give direct proofs, referring to any standard algebra textbook (e.g., Lang) for more general statements/results. [7]

Let k be an algebraically closed field. Let V be a nonzero finite-dimensional vector space over k. The space $\mathrm{End}(V)$ of k-linear endomorphisms of V is a k-algebra. Let G be any subgroup of $\mathrm{GL}(V)$. Let R denote the k-subalgebra of $\mathrm{End}(V)$ generated by G. In fact, since G is a group, the algebra R is spanned by G as a vector space, i.e., R is the set of finite sums $\sum_{g \in G} c_g g \in \mathrm{End}(V)$ (where the coefficients $c_g \in k$ vanish for all but finitely many $g \in G$). We henceforth impose the following assumption:

$$V \text{ is } G\text{-irreducible},$$

i.e., $\{0\}$ and V are the only G-invariant subspaces of V.

Theorem 6.1. $R = \mathrm{End}(V)$.

Example 6.2. In the special case that the group G is finite, the theorem follows from a basic result in representation theory: the matrix coefficients of an irreducible representation of a finite group are linearly independent, hence the image of such a representation spans the space of endomorphisms.

We turn to the proof. Let

$$D := \{x \in \mathrm{End}(V) : xg = gx \text{ for all } g \in G\}$$

denote the space of all G-*equivariant* operators on V, i.e., those that commute with every element of G. Since R is spanned by G, we see that D is also the space of all operators on V

[7] End of lecture #7, 11 Mar 2020

that commute with every element of R. It's clear that D contains the subspace $k \subseteq \operatorname{End}(V)$ of scalar operators (i.e., multiples of the identity), since the latter commute with everything. Thus D is a k-subalgebra of $\operatorname{End}(V)$.

Lemma 6.3 (Schur's lemma)**.** *We have $D = k$, i.e., every element of D is a scalar operator.*

Proof. Let $x \in D$. Since V is nonzero and k is algebraically closed, the operator $x : V \to V$ has a nontrivial eigenspace V_λ. Since x commutes with the action of G, the subspace V_λ of G is G-invariant: indeed, for $v \in V_\lambda$, we have $xgv = gxv = g\lambda v = \lambda gv$, hence $gv \in V_\lambda$. Since V is G-irreducible and V_λ is nontrivial, we must have $V_\lambda = V$, i.e., $x = \lambda \in k$, as required. □

Now let $n \in \mathbb{Z}_{\geq 1}$, and write V^n for the direct sum of n copies of the vector space V. Given $v = (v_1, \dots, v_n) \in V^n$, we write

$$Rv := \{(rv_1, \dots, rv_n) : r \in R\} = \operatorname{span}\{(gv_1, \dots, gv_n) : g \in G\}.$$

Lemma 6.4. *Let $v = (v_1, \dots, v_n) \in V^n$. If $Rv \neq V^n$, then the v_j are linearly dependent.*

Proof. For clarity, we write $V^n = V_1 \oplus \cdots \oplus V_n$, where the $V_j \cong V$ denote isomorphic copies of V, thus $V_j \leq V^n$ is the jth direct summand. Suppose $Rv \neq V^n$. We may find a largest possible integer J in the range $\{1, \dots, n\}$ with the property that the subspace $U := \sum_{1 \leq j < J} V_j + Rv$ of V^n satisfies $U \neq V^n$. (For instance, if $V_1 + Rv = V^n$, then $J = 1$; if $V_1 + Rv \neq V^n$ but $V_1 + V_2 + Rv = V^n$, then $J = 2$.) Then $V_J + U = V^n$. Since $U \neq V^n$, we know that U does not contain V_J. Hence the intersection $V_J \cap U$ is a proper subspace of V_J. Letting G act on V^n diagonally, it's clear from the definition that U is a G-invariant subspace. Hence $V_J \cap U$ of V_J is G-invariant. Since V_J is G-irreducible, it follows that $V_J \cap U = \{0\}$. Thus the composition

$$V \cong V_J \hookrightarrow V^n \to V^n/U \tag{6.1}$$

is both surjective (because $V_J + U = V^n$) and injective (because $V_J \cap U = \{0\}$), hence is an isomorphism. Moreover, it is G-equivariant, hence likewise its inverse is G-equivariant. Let $\iota : V^n/U \to V$ denote the inverse isomorphism. Let $\ell : V^n \to V$ denote the pullback of ι under the quotient map $V^n \to V^n/U$. Then ℓ is G-equivariant. Let ℓ_j denote the restriction of ℓ to $V_j \leq V^n$. Then ℓ_j defines a G-equivariant operator $V \to V$. By lemma 6.3, we see that $\ell_j = t_j \in k$ is a scalar operator. Since ℓ is an isomorphism, we know that $(t_1, \dots, t_n)$ is nonzero. Since ℓ vanishes on U and hence also on Rv, we obtain the required nontrivial linear relation $\sum t_j v_j = 0$. □

Proof of Theorem 6.1. Set $n := \dim(V)$. Let $e_1, \dots, e_n$ be a basis of V. Set $e := (e_1, \dots, e_n) \in V^n$. Since the e_j are linearly independent, we see by lemma 6.4 that $Re = V^n$. Thus for each $x \in \operatorname{End}(V)$ we may find $r \in R$ so that $re_j = xe_j$ for all j, which implies that $x = r$. Thus $R = \operatorname{End}(V)$. □

7. Unipotent groups

Let V be a finite-dimensional vector space. Set $n := \dim(V)$.

Definition 7.1. A *complete flag* in V is a maximal strictly increasing chain of subspaces, i.e., a chain of the form

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V.$$

Note that every complete flag $V_0 \subset \cdots \subset V_n$ may be expressed as $V_j = \operatorname{span}\{e_1, \ldots, e_j\}$ for some basis $e_1, \ldots, e_n$ of V. A linear operator on V is upper-triangular with respect to such a basis precisely when it leaves invariant each subspace of the corresponding complete flag.

Definition 7.2. A *unipotent subgroup* G of $\mathrm{GL}(V)$ is a subgroup that consists of unipotent elements.

Lemma 7.3. *Let G be a unipotent subgroup of $\mathrm{GL}(V)$. There is a complete flag in V consisting of G-invariant subspaces.*

Corollary 7.4. *Every unipotent subgroup of $\mathrm{GL}_n(k)$ is conjugate to a subgroup of the group U_n of upper-triangular unipotent matrices (i.e., upper-triangular matries with 1's on the diagonal).*

Recall that to a group G one attaches the commutator subgroup

$$G' := (G, G) := \langle (g, h) := ghg^{-1}h^{-1} : g, h \in G \rangle$$

as well as the series of subgroups $G^{(n)}, G_n$ by the rules

$$G^{(0)} := G_0 := G, G_{n+1} := (G, G_n), \quad G^{(n+1)} := (G^{(n)}, G^{(n)}).$$

Recall that G is *nilpotent* (resp. *solvable*) if $G_n = \{0\}$ (resp. if $G^{(n)} = \{0\}$) for some n. Nilpotent groups are solvable.

Corollary 7.5. *Any unipotent subgroup of $\mathrm{GL}(V)$ is nilpotent.*

(The following was not stated in lecture, but seems worth recording.)

Lemma 7.6. *Let G be an algebraic group. The following are equivalent:*

- *(i) Each $g \in G$ is unipotent.*
- *(ii) Every irreducible representation of G is one-dimensional and trivial.*
- *(iii) Each nonzero finite-dimensional V of G contains a nonzero vector fixed by G.*
- *(iv) Each finite-dimensional representation V of G admits a basis with respect to which G is represented by strictly upper-triangular matrices.*
- *(v) G is isomorphic to an algebraic subgroup of the group U_n of strictly upper-triangular elements of $\mathrm{GL}_n(k)$ for some n.*

Definition 7.7. We say that an algebraic group G is *unipotent* if it satisfies the equivalent conditions of the lemma.

[8]

8. Diagonalizable groups and tori

A change of notation: $\mathcal{O}(X)$ will henceforth denote the k-algebra of regular functions on an algebraic set X, $k[M]$ will denote the group algebra attached to a group M (see below).

Lemma 8.1. *Let S be a pairwise commutative subset of* $\mathrm{End}(V)$.

Then V admits a complete S-invariant flag. In other words, there is a basis with respect to which S is upper-triangular.

If moreover each element of S is semisimple, then V admits a basis of eigenvectors. In other words, there is a basis with respect to which S is diagonal.

In this section we write $\mathbb{G}_m$ for the multiplicative group over our given algebraically closed field k.

Definition 8.2. An algebraic group G is said to be *torus* if, for some $n \in \mathbb{Z}_{\geq 0}$, G is isomorphic the nth power $\mathbb{G}_m^n$ of the multiplicative group, or equivalently, to the algebraic group consisting of all diagonal elements in $\mathrm{GL}_n(k)$.

Corollary 8.3. *Let G be a commutative algebraic group. The following are equivalent:*

(i) Each $g \in G$ is semisimple.
(ii) Each finite-dimensional representation V of G admits a basis of G-eigenvectors, i.e., a basis with respect to which $r_V(G)$ consists of diagonal matrices.
(iii) G is isomorphic to an algebraic subgroup of a torus.

Definition 8.4. We say that an algebraic group G is *diagonalizable* if it satisfies the equivalent conditions stated above.

Definition 8.5. Let G be an algebraic group. A *character* of G is a morphism of algebraic groups

$$\chi : G \to \mathbb{G}_m.$$

Lemma 8.6. *The characters of G correspond to those $\chi \in \mathcal{O}(G)^\times$ for which $\Delta(\chi) = \chi \otimes \chi$.*

The set of characters of G forms an abelian group denoted $\mathfrak{X}(G)$. A morphism $f : G \to H$ of algebraic groups defines a morphism $f^* : \mathfrak{X}(H) \to \mathfrak{X}(G)$ of abelian groups. We obtain a functor from algebraic groups to abelian groups.

Lemma 8.7. *Let $T \cong \mathbb{G}_m^n$ be a torus. Then*

$$\mathfrak{X}(T) \cong \mathbb{Z}^n$$

$$\chi_m \leftrightarrow m$$

$$\chi_m(y) = y_1^{m_1} \cdots y_n^{m_n}.$$

[8] End of lecture #8, 16 Mar 2020

Lemma 8.8. *Let G be a diagonalizable algebraic group. Then the character group $\mathfrak{X}(G)$ defines a basis of the k-vector space $\mathcal{O}(G)$.*

Proof. Independence: just use that characters of any group are independent.

Spanning: reduce to the case of a torus, then appeal to Lemma 8.7. ☐

Definition 8.9. Let M be an abelian group. The *group algebra* $k[M]$ attached to M is the k-vector space with M, consisting of formal finite sums $\sum_{m \in M} c_m m$, with the multiplication law extending that on M, thus $\sum_{m \in M} a_m m \sum_{n \in M} b_n n = \sum_{m,n \in M} a_m b_n mn$. There is a unique Hopf algebra structure on $k[M]$ for which $\Delta(m) = m \otimes m$ for all $m \in M$.

Corollary 8.10. *The coordinate ring $\mathcal{O}(G)$ of a diagonalizable algebraic group identifies (as a Hopf algebra) with the group algebra $k[\mathfrak{X}(G)]$ attached to the character group $\mathfrak{X}(G)$.*

Lemma 8.11. *Let M_1, M_2 be abelian groups. Extension defines a bijection*

$$\{\textit{morphisms of abelian groups } M_1 \to M_2\} \cong \{\textit{morphisms of Hopf algebras } k[M_1] \to k[M_2]\}.$$

Corollary 8.12. *For diagonalizable algebraic groups G and H, the map $f \mapsto f^*$ defines a bijection*

$$\operatorname{Hom}(G, H) \cong \operatorname{Hom}(\mathfrak{X}(H), \mathfrak{X}(G)),$$

where the first Hom *is of algebraic groups and the second is of abelian groups.*

Lemma 8.13. *The character group of a diagonalizable algebraic group is a finitely-generated abelian group with no torsion elements of order* $\operatorname{char}(k)$.

Conversely, let M be a finitely-generated abelian group with no torsion elements of order $\operatorname{char}(k)$. *Then M is the character group of some diagonalizable algebraic group.*

Proof. The key point is that $k^\times$ has no torsion elements of order $\operatorname{char}(k)$, but does have torsion elements of any order prime to $\operatorname{char}(k)$. ☐

Theorem 8.14. *Taking character groups defines an equivalence of categories*

$$\{\textit{diagonalizable algebraic group}\} \cong$$
$$\{\textit{finitely-generated abelian groups with no torsion elements of order } \operatorname{char}(k)\}$$

and also

$$\{\textit{tori}\} \cong \{\textit{finitely-generated free abelian groups}\}.$$

9. Trigonalizable groups: definition

We say that a representation $r : G \to \mathrm{GL}(V)$ of a group G is *trigonalizable* if V admits a basis with respect to which G is upper-triangular. We say that a subgroup G of $\mathrm{GL}(V)$ is trigonalizable if the identity representation is.

Lemma 9.1. *Let G be an algebraic group. The following conditions are equivalent:*

(i) Every irreducible representation of G is one-dimensional.
(ii) Every finite-dimensional representation of G is trigonalizable.

(iii) G is isomorphic to an algebraic subgroup of the upper-triangular subgroup B_n of $\mathrm{GL}_n(k)$ for some n.

Proof. (i) implies (ii): Induct on the dimension of V.

(ii) implies (iii): Consider any faithful representation.

(iii) implies (i): Let $A_n, U_n \leq B_n$ denote the subgroups consisting of diagonal and strictly upper-triangular elements, respectively. Set $U := G \cap U_n$. Then U is a normal unipotent algebraic subgroup of G. Let $r : G \to \mathrm{GL}(V)$ be an irreducible representation of G. Since V is nonzero, the subspace V^U of U-fixed vectors is nonzero. Since U is normal, that subspace V^U is G-invariant. Since V is irreducible, we have $V = V^U$. Thus r descends to a representation of the abstract group G/U. The latter is isomorphic to to a subgroup of A_n, so the image of r is commutative. On the other hand, for $g \in G$ with Jordan decomposition $g = g_s g_u$, we have $r(g) = r(g_s)r(g_u) = r(g_s)$. Thus $r(G)$ is a commutative set consisting of semisimple operators. Thus V admits a basis of G-eigenvectors. Since V is irreducible we conclude as required that $\dim V = 1$. □

Definition 9.2. We say that an algebraic group G is *trigonalizable* if it satisfies the equivalent conditions of the lemma.

10. Jordan decomposition of commutative groups

Let G be any algebraic group. Write G_s (resp. G_u) for the set of semisimple (resp. unipotent) elements of G. The set G_u is always algebraic, but G_s need not be algebraic in general. Neither set is a subgroup in general.

Example 10.1. If $G = \mathrm{SL}_2(k)$, then $\begin{pmatrix} \lambda & 1 \\ & \lambda^{-1} \end{pmatrix}$ belongs to G_s if and only if $\lambda^2 \neq 1$, which is not an algebraic condition on λ. Thus G_s is not closed.

Theorem 10.2. *Let G be a commutative algebraic group. Then G is trigonalizable. Moreover, G_s and G_u are algebraic subgroups of G, the multiplication map $G_s \times G_u \to G$ is an isomorphism of algebraic groups, and every finite-dimensional representation V of G admits a basis with respect to which G_s (resp. G_u) is given by diagonal (resp. strictly upper-triangular) matrices.*

Given an element g of an algebraic group, let us temporarily write $G(g)$ for the Zariski closure of the group $\{g^n : n \in \mathbb{Z}\}$ generated by g. By a homework problem, $G(g)$ is an algebraic group.

Corollary 10.3. *We have $G(g)_s = G(g_s)$ and $G(g)_u = G(g_u)$, hence $G(g) \cong G(g_s) \times G(g_u)$.*

11. Connected solvable groups are trigonalizable

Lemma 11.1. *Let G be a subgroup of $\mathrm{GL}(V)$. If G is connected, then so is $G' := (G, G)$.*

Theorem 11.2. *Let G be a connected solvable subgroup of some general linear group. Then G is trigonalizable.*

Proof. We may assume that G is algebraic. It suffices to show that every irreducible representation $r : G \to \mathrm{GL}(V)$ is one-dimensional. We've seen this already when G is commutative. Assume otherwise that G is non-commutative, i.e., that $G' := (G, G)$ is non-trivial. Since G is solvable, we have $G' \neq G$. Since G is connected, so is G'. Thus G' is a connected normal subgroup of G. We may inductively assume that G' has an eigenspace in V. Since G' is normal, the group G acts on the eigenspaces for G'. Since V is assumed irreducible, we may write $V = \oplus V_\chi$, where χ runs over characters of G'.

We claim that any such eigenspace V_χ is G-invariant. To see this, observe first that only finitely many characters appear in the decomposition of V; deduce that for each $h \in G'$, the map $G \to \mathrm{GL}(V)$ given by $g \mapsto r(g)r(h)r(g)^{-1}$ has finite image; deduce from the connectedness of G that such maps are constant; deduce finally that G' is central, hence that G stabilizes V_χ.

Thus from the irreducibility, we deduce that V is itself a G'-eigenspace, i.e., $V = V_\chi$ for some character χ of G'. Since G' is generated by commutators, we have $\det(r(G')) = \{1\}$. Thus, writing $d := \dim(V)$, we deduce that $r(G')$ is contained in the group μ_d of dth roots of unity. Now, the map r is algebraic, and G' is connected, hence the image $r(G')$ is connected. Since μ_d is finite, this implies that $r(G') = \{1\}$. So r is really a representation of the quotient group G/G'. But that group is commutative, and we've seen (Lemma 8.1) that commutative subsets of $\mathrm{GL}(V)$ may be upper-triangularized. Since V is irreducible, we conclude that $\dim V = 1$. □

Corollary 11.3. *Let G be a connected algebraic group. Then G is trigonalizable iff G is solvable.*

12. Semisimple elements of connected nilpotent groups are central

We record the first lemma for motivational purposes only.

Lemma 12.1. *Let V be a finite-dimensional vector space and $\mathfrak{g} \subseteq \mathrm{End}(V)$ a Lie subalgebra. Suppose that $\mathfrak{g}$ is nilpotent. Then any semisimple element $x \in \mathfrak{g}$ is central in $\mathfrak{g}$.*

Proof. Since $\mathfrak{g}$ is nilpotent, the operator $\mathrm{ad}(x)$ is nilpotent. Since x is semisimple, the operator $\mathrm{ad}(x)$ is semisimple. Thus $\mathrm{ad}(x) = 0$, i.e., x is central. □

Lemma 12.2. *Let G be a connected nilpotent algebraic group. Let $g \in G$ be semisimple. Then $g \in Z(G)$.*

Proof. We give the proof first in the case $k = \mathbb{C}$. In that case, since G is connected, it suffices to check that $\mathrm{Ad}(g)$ is trivial. We may assume that $G \subseteq \mathrm{GL}(V)$. Since g is semisimple, we may write it as the exponential of some semisimple Lie algebra element $x \in \mathrm{Lie}(G)$. It then suffices to show that x is central in $\mathfrak{g}$. For this we appeal to the preceeding lemma.

We now prove the result for general k. Let $h \in G$, and suppose that g and h do not commute. Choose a faithful representation $G \hookrightarrow \mathrm{GL}(V)$. Since G is solvable, we may find a complete G-invariant flag $V_0 \subset \cdots \subset V_n$. The restrictions of g and h to V_i commute if $i = 0$ but do not commute if $i = n$. We may thus find $i \in \{0..n-1\}$ so that g and h commute on V_i, but not on V_{i+1}. Let $a, b \in \mathrm{GL}(V_i)$ denote the restrictions of g, h to V_i. Then a is semisimple, and a and b commute. Since g is semisimple, we may write $V_{i+1} = V_i \oplus \langle v \rangle$ with v an eigenvector of g. By replacing g and h by scalar multiples, we may assume that they stabilize v. We may then write

$$g \sim \begin{pmatrix} a & \\ & 1 \end{pmatrix}, \quad h \sim \begin{pmatrix} b & c \\ & 1 \end{pmatrix}$$

for some $c \in V_i$. Then

$$ghg^{-1} \sim \begin{pmatrix} b & ac \\ & 1 \end{pmatrix}. \tag{12.1}$$

Our hypothesis that g and h do not commute says that $ac \neq a$, i.e., that $c \notin \ker(a-1)$. Set

$$h_1 := h^{-1} g h g^{-1}. \tag{12.2}$$

We have

$$h = \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix} \begin{pmatrix} b & \\ & 1 \end{pmatrix}, \quad h^{-1} = \begin{pmatrix} b^{-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix}, \tag{12.3}$$

so

$$\begin{aligned} h_1 &:= h^{-1} g h g^{-1} \\ &= \begin{pmatrix} b^{-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix} \begin{pmatrix} b & ac \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & b^{-1}(a-1)c \\ & 1 \end{pmatrix}. \end{aligned}$$

We claim that h_1 does not commute with g. Indeed, by the same calculation as for h, this boils down to asking that $b^{-1}(a-1)c \notin \ker(a-1)$. But

$$(a-1)b^{-1}(a-1)c = b^{-1}(a-1)^2 c,$$

which vanishes precisely when $c \in \ker((a-1)^2)$. But since $a-1$ is semisimple, we have $\ker((a-1)^2) = \ker(a-1)$. Hence the claim.

We inductively define $h_{i+1} := h_i^{-1} g h_i g^{-1}$. Each of these elements is nontrivial. But h_i belongs to the central series G_i of G. Since G is nilpotent, this gives the required contradiction. □

13. Algebro-geometric preliminaries

We spent several lectures introducing/reviewing some basics from algebraic geometric concerning quasi-projective varieties (assumed irreducible for our purposes), dimension, and images of morphisms. We followed Chapter 3 of Szamuely's notes closely. We record some of the highlights here.

We explained how Grassmannians and flag varieties may be regarded as projective varieties. Beyond the course references, notes such as `https://personal-homepages.mis.mpg.de/michalek/may08.pdf` may be useful to consult.

We took for granted the following facts from algebraic geometry:

Theorem 13.1. *Let $\phi : X \to Y$ be a morphism of quasi-projective varieties whose image is dense. Then its image contains a dense open subset of Y.*

We used this to establish some basics concerning orbits of algebraic group actions:

Theorem 13.2. *Let G be an algebraic group acting on a quasi-projective variety X. Then every orbit of this action is locally closed (i.e., open in its closure), hence may be regarded as a quasi-projective variety. The orbits of minimal dimension are closed.*

We stated without proof the following two assertions from algebraic geometry, but explained their relationship:

Theorem 13.3. *Let X be a projective variety, and let Y be a quasi-projective variety.*

1. *The projection map $X \times Y \to Y$ is closed, i.e., maps closed sets to closed sets.*
2. *The image of any morphism $X \to Y$ is closed.*

We deduced the general case of the Borel fixed point theorem:

Theorem 13.4. *Let G be a connected solvable algebraic group. Let X be a projective variety, equipped with an action of G. Then this action has a fixed point, i.e., an element $x \in X$ for which $gx = x$ for all $g \in G$.*

14. Homogeneous spaces and quotients

Definition 14.1. By a *homogeneous space* X for a connected algebraic group G we mean a quasi-projective variety equipped with a *transitive* action of G. (For disconnected G, we should, strictly speaking, X to be a finite disjoint union of (irreducible) quasi-projective varieties, but we will often ignore this detail.)

Let H be an algebraic subgroup of an algebraic group G. Consider the category of pairs (X, ρ) consisting of a quasi-projective variety X equipped with a morphism $\rho : G \to X$ that is constant on the left cosets gH of H. An arrow $(X, \rho) \to (X', \rho')$ is a morphism $\phi : X \to X'$ such that $\phi \circ \rho = \rho'$.

Definition 14.2. A *quotient* of G by H is an initial object in this category, i.e., a pair (X, ρ) as above such that for every pair (X', ρ') as above, there is a unique morphism $\phi : X \to X'$ so that $\phi \circ \rho = \rho'$.

The usual arguments involving universal properties show that if such a quotient exists, then it is unique up to unique isomorphism.

Theorem 14.3. *Such a quotient (X, ρ) always exists. Moreover, it is a G-homogeneous space containing a point P having stabilizer H, and with ρ given by the orbit map $g \mapsto gP$.*

A first step in the direction of the proof is the following:

Proposition 14.4. *For any algebraic subgroup H of an algebraic group G there is a G-homogeneous space X and a point $P \in X$ so that $H = \mathrm{Stab}_G(P)$.*

Proof. Direct consequence of Theorem 3.11 and Theorem 13.2: we choose a finite dimensional representation V of G and a line $L \subseteq V$ so that H is the stabilizer of L, hence also the stabilizer of the point P corresponding to L in the projectivization $\mathbb{P}(V)$, and take for X the orbit $\mathcal{O} := G \cdot P \subseteq \mathbb{P}(V)$. □

We sketched the main step in passing from Proposition 14.4 to Theorem 14.3 in the case that k has characteristic zero. (We followed Szamuely's notes, which seem a bit misleading in the finite characteristic case.) The arguments give the following stronger conclusion:

Theorem 14.5. *Suppose that k has characteristic zero. Let H be an algebraic subgroup of an algebraic group G. Let X be a G-homogeneous space and $P \in X$ a point whose stabilizer is H. Let $\rho : G \to X$ denote the orbit map $g \mapsto gP$. Then (X, ρ) is a quotient of G.*

The key step in verifying this was the following:

Lemma 14.6. *Let conditions be as in Theorem 14.5. Let $U \subseteq X$ be an open set, with preimage $\rho^{-1}U \subseteq G$. Let $f \in \mathcal{O}(\rho^{-1}U)$ be a regular function on that preimage that is constant on left H-cosets. Let $\bar{f} : U \to k$ be the unique map of sets for which $f = \bar{f} \circ \rho$. Then $\bar{f}$ is a regular function on U.*

We argued first that $\bar{f}$ defines a *rational* function on U using the following:

Lemma 14.7. *Suppose that k has characteristic zero. Then any injective morphism of quasi-projective varieties $X \to Y$ with dense image is birational, i.e., induces, via pullback, an isomorphism of function fields $k(Y) \cong k(X)$.*

The next part of the argument (only briefly sketched in lecture) was that X, being a homogeneous space for G, is smooth, and on a smooth quasi-projective variety, every rational function that fails to be regular has a "pole."

We gave a simple example ($G = k \hookrightarrow X = k$ via $g \cdot x := x + g^p$, $H = \{1\}$) showing that the statements of Theorem 14.5 and Lemma 14.7 are false in the positive characteristic case $p > 0$; in that case, one needs to assume further properties of the action $G \to X$, e.g., that the induced action on Zariski tangent spaces be submersive (which fails in the indicated example because $d(g^p) = pg^{p-1}dg = 0$ in characteristic p) or that H describe the stabilizer of P not only at the level of k-points, but also at the level of R-points for every finitely-generated k-algebra R (which fails in the indicated example because H has no nontrivial k-points but plenty of nontrivial $k[\varepsilon]/(\varepsilon^2)$-points). For a careful treatment in the general characteristic case of Theorem 14.3, we refer to sections 11 and 12 of Humphrey's textbook and also Section 7e (and references) of Milne's textbook (listed on the course homepage).

15. Borel and parabolic subgroups

We followed Szamuely, section 23.

16. Reductivity

We followed Szamuely, sections 23 and 20.

17. Union of Borel subgroups

We followed Szamuely, Lemma 26.3 (1) and Onishchik–Vinberg, Section 3.3.9.

18. Splitting solvable groups

We followed Onishchik–Vinberg, Section 3.3.7.

19. Basics on maximal tori

We followed some mixture of Szamuely and Onishchik–Vinberg.

Definition 19.1. Let G be an algebraic group. A *maximal torus* T in G is a torus in G that is not contained in any strictly larger torus. (Or should we define it to be one of maximal dimension?)

Here are the main results:

Theorem 19.2. *All maximal tori in G are conjugate.*

Theorem 19.3. *If G is connected, then every semisimple element is contained in some maximal torus.*

We can reduce the proofs of these assertions to those of the following, which are of independent interest:

Theorem 19.4. *If G is connected, then it is the union of its Borel subgroups. In other words, if B is any Borel subgroup, then $G = \cup_{g \in G} gBg^{-1}$.*

Theorem 19.5. *Let B be a connected solvable algebraic group. Let $U := G_u$ denote its unipotent radical Then the quotient B/U is a torus. Moreover, there is a torus $T \leq B$ for which the composition $T \hookrightarrow B \twoheadrightarrow B/U$ is an isomorphism. Such a torus T is a maximal torus, and G is the semidirect product $U \rtimes T$.*

Example 19.6. In general, there are many such tori T in B; they are not at all canonical. Consider the case

$$B = \begin{pmatrix} * & * \\ & * \end{pmatrix}, \quad U = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}.$$

We can take $T \leq B$ to be the subgroup of diagonal matrices, or any conjugate of it. For instance, conjugating the group of diagonal matrices by the element $\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}$ gives the group

$$T = \left\{ \begin{pmatrix} a & (b-a)u \\ & b \end{pmatrix} : a, b \in k^\times \right\}.$$

Finally:

Proposition 19.7. *Let B be a connected solvable algebraic group, decomposed as $B = U \rtimes T$ as above. Then every semisimple element of B is conjugate to an element of T.*

Theorem 19.8. *Let G be a connected algebraic group, and let $S \leq G$ be a torus. Then $Z_G(S)$ is connected.*

We focused on the characteristic zero case, which is simpler because unipotent subgroups are automatically connected, and proved the following results as intermediary steps.

Theorem 19.9. *Suppose k has characteristic zero. Then every unipotent algebraic group is connectedness. In fact, if $g \in \mathrm{GL}_n(k)$ is unipotent, then the algebraic group $G(g)$ generated by g consists is given by $G(g) = \{\exp(t \log g) : t \in \mathbb{C}\}$, and is isomorphic to the additive group k. (Here* exp *and* log *are defined by truncating the infinite series definitions.)*

Theorem 19.10. *Suppose k has characteristic zero. Then any torus T admits a* generator *t, i.e., an element $t \in T$ for which $\overline{\langle t \rangle} = T$.*

We followed roughly the following outline:

- For the union of Borel subgroups, we first gave the general proof that the union is closed. We then proved for $k = \mathbb{C}$ that it is dense. For this we needed to know that any Borel subalgebra of $\mathfrak{g}$ is the Lie algebra of a Borel subgroup of G; see Onishchik–Vinberg, Problem 3.3.8.
- For the splitting of solvable algebraic groups and semidirect product structure, we restricted to characteristic zero and used that tori have generators and that bijective morphisms are isomorphisms: see Onishchik–Vinberg, 3.1.4, Theorem 6.
- For the theorem that all semisimple elements in a connected group are contained in a torus (hence in a maximal torus), we reduced by considering Borel subgroups to showing for a connected solvable group $B = U \rtimes T$ that any semisimple element is conjugate to some element of T. For this we needed a couple lemmas, both specific to characteristic zero.
  - Unipotent algebraic groups are connected, and the commutative ones are actually isomorphic to vector spaces. We explained this using the (formal) exponential map.
  - There exists V, a unipotent normal algebraic subgroup of B, so that V has codimension one in U. We then inducted on $\dim(U)$.
- To verify that all maximal tori in a connected group are conjugate, we used generators.

20. Borel subgroups are self-normalizing

Lemma 20.1. *Let G be a connected algebraic group, let B be a Borel subgroup, and let S be a torus contained in B. Then $Z_B(S)$ is a Borel subgroup of $Z_G(S)$.*

Proof. Let U denote the unipotent radical of B. It is enough to establish the identity

$$Z_G(S)B = \{y \in G : y^{-1}sy \in sU \text{ for all } s \in S\}. \tag{20.1}$$

Indeed, let us assume this for the moment. Since the RHS of (20.1) is closed, it follows that $Z_G(S)B$ is closed. Since the projection $G \to G/B$ is open and surjective, it follows that the image of $Z_G(S)$ in G/B is closed. That image identifies with the quotient $Z_G(S)/Z_B(S)$. We have seen $Z_B(S)$ is connected. It is thus a parabolic subgroup of $Z_G(S)$. It is also solvable, hence is a Borel subgroup of $Z_G(S)$, as required.

The containment "$\subseteq$" in (20.1) follows from the fact that U is normal in B and the quotient B/U (a torus) is commutative. Conversely, suppose $y \in G$ satisfies $y^{-1}sy \in sU$ for all $s \in S$. In particular, $y^{-1}Sy \subseteq SU$. By conjugacy of maximal tori, we may thus find $b \in SU$ so that $by^{-1}Syb^{-1} = S$. Since B/U is commutative, we then have for all $s \in S$ that $by^{-1}syb^{-1} \in sU \cap S = \{s\}$, and thus $yb^{-1} \in Z_G(S)$, i.e., $y \in Z_G(S)B$, as required.

□

Theorem 20.2. *Let B be a Borel subgroup of a connected algebraic group G. Then $N_G(B) = G$.*

Proof. We induct on the dimension of G. In the one-dimensional case, we have seen that G is commutative, so $B = G$ and $N_G(B) = G = B$. We turn to the general case.

Suppose that $x \in G$ normalizes B. Our goal is to show that $x \in B$. Let T be a maximal torus contained in B. Since B acts transitively on its maximal tori, we may assume without loss of generality that x normalizes T. Consider, then, the endomorphism $\rho : T \to T$ givenby

$$\rho(t) = xtx^{-1}t^{-1}. \tag{20.2}$$

We argue separately according as ρ is surjective or not.

If ρ is not surjective, then we may apply our induction hypothesis, as follows. Since T is irreducible, we know that the image of ρ has dimension less than the dimension of T. It follows that $S := \ker(\rho)^0$ has positive dimension, and is thus a nontrivial torus. By construction, x centralizes S, hence normalizes $Z_B(S)$. By Lemma 20.1, $Z_B(S)$ is a Borel subgroup of the (connected) group $Z_G(S)$. Thus if $Z_G(S) \neq G$, then our inductive hypothesis gives $x \in Z_B(S) \subseteq B$, as required. If instead $Z_G(S) = G$, then the subgroup S of G is central, hence normal, and the quotient B/S is a Borel subgroup of G/S, so we may include by applying our inductive hypothesis.

It remains to address to the case that ρ is not surjective. Set $H := N_G(B)$. Then

$$T = \{xtx^{-1}t^{-1} : t \in T\} \subseteq (H, H). \tag{20.3}$$

Choose a linear representation $G \hookrightarrow \mathrm{GL}(V)$ and a line $L \subseteq V$ so that $H = \{g \in G : gL = L\}$. The induced action of H on L is described by a map

$$\gamma : H \to \mathrm{GL}(L) = \mathbb{G}_m. \tag{20.4}$$

Since every element of $\mathbb{G}_m$ is semisimple, we see that γ resstricts trivially to the subset H_u of unipotent elements. Since $\mathbb{G}_m$ is commutative, we see that γ restricts trivially to the commutator subgroup (H, H) of H; in particular, γ restricts trivially to T. Since $B = B_uT \subseteq H_uT$, it follows that γ restricts trivially to B. By choosing a basis element v of L, we obtain a well-defined morphism

$$G/B \to V$$
$$gB \mapsto gv.$$

Since G/B is projective and connected while V is affine, this morphism is constant. Thus $G = H$, i.e., B is normal in G. Since G is the union of the conjugates of B, we conclude as required that $B = G = N_G(B)$. □

Corollary 20.3. *The map $gB \mapsto gBg^{-1}$ identifies G/B with the set of Borel subgroups in G.*

21. The Borel subgroups containing a given maximal torus

Let G be a connected algebraic group and $T \leq G$ a maximal torus. Write $\mathcal{B}^T$ for the set of Borel subgroups B of G that contain T. Observe that $N_G(T)$ acts on $\mathcal{B}^T$ by conjugation.

Lemma 21.1. *The action of $Z_G(T)$ on $\mathcal{B}^T$ is trivial.*

Proof. We have seen that $Z_G(T)$ is connected. Since T is central in $Z_G(T)$, we see (by an earlier result) that $Z_G(T)$ is nilpotent. We may thus find a Borel subgroup B_0 in G that contains $Z_G(T)$.

Now let $B \in \mathcal{B}^T$. We must verify that $Z_G(T)$ normalizes B. Since B is self-normalizing, it is equivalent to verify that $Z_G(T) \subseteq B$.

To that end, choose $g \in G$ such that $B = gB_0g^{-1}$. By the conjugacy of maximal tori in B, we may assume that g normalizes T. Then by elementary group theory, it follows that g normalizes also $Z_G(T)$, hence

$$g^{-1}Z_G(T)g = Z_G(T) \subseteq B,$$

and so

$$Z_G(T) \subseteq gB_0g^{-1} = B,$$

as required. □

It follows that the action of $N_G(T)$ on $\mathcal{B}^T$ descends to an action of the Weyl group $W = W(G,T) = N_G(T)/Z_G(T)$.

Theorem 21.2. *W acts simply-transitively on $\mathcal{B}^T$.*

Proof. The transitivity follows as above. For the freeness of the action, let $B \in \mathcal{B}^T$. We must check that $N_G(T) \cap N_G(B) \subseteq Z_G(T)$, i.e., that

$$N_B(T) \subseteq Z_G(T).$$

Set $U := B_u$. Since $B = TU$, it suffices to show that

$$U \cap N_B(T) \subseteq Z_B(T).$$

Let $n \in U \cap N_B(T)$ and $t \in T$. Write $t' := ntn^{-1} \in T$. By reducing modulo U and using that the composition $T \hookrightarrow B \twoheadrightarrow B/U$ is an isomorphism, we see that $t' = t$. Thus $n \in Z_B(T)$, as required. □

Remark 21.3. As mentioned in lecture for those who have previously studied (e.g.) compact Lie groups, $\mathcal{B}^T$ corresponds to the set of Weyl chambers for T, and the results proved above translate to the fact that the Weyl group acts simply-transitively on the set of chambers.

22. Producing elements of the Weyl group

Proposition 22.1. *Let G be a connected non-solvable algebraic group. Let B be a Borel subgroup, T a maximal torus, and W the associated Weyl group. Then $\#W \geq 2$, with equality iff $\dim(G/B) = 1$.*

Proof. For lack of time, we only sketched the proof, referring to [Szamuely, §29] for details. A Borel subgroup gBg^{-1} of G contains T iff the coset $gB \in G/B$ is fixed by T, so the first inequality says that the action of T on G/B has at least two fixed points. Consider for simplicity the case that $T = \mathbb{G}_m$. We obtain then a nontrivial orbit map $\mathbb{G}_m \to G/B$ which, by the projectivity of G/B, extends to a morphism $\mathbb{P}^1 \to G/B$. The main point is then to verify that the images in G/B of $0, \infty \in \mathbb{P}^1$ are distinct fixed points. This gives that $\#W \geq 2$.

A more elaborate argument shows that if $\dim(G/B) \geq 2$, then the action of T on G/B has at least three fixed points. From this it follows that in the equality case $\#W = 2$, we have $\dim(G/B) = 1$, as claimed. □

Proposition 22.2. *Let G be a connected algebraic group of semisimple rank 1. There is a surjective morphism $\rho : G \to \mathrm{PGL}_2(k)$ with $\ker(\rho)^0 = R(G)$. In particular, if G is semisimple, then $\ker(\rho)$ is finite.*

If moreover G is reductive, then $\ker(\rho)$ is diagonalizable.

Example 22.3. The proposition applies to the following subgroups of GL_3:

$$\begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}, \quad \begin{pmatrix} \mathrm{SL}_2 & \\ & * \end{pmatrix}.$$

Proof. We again only sketch the proof. By modding out by $R(G)$ and carefully checking that the hypotheses remain valid, we may reduce to the case that G is semisimple and of rank 1. In particular, G is non-solvable. We use the preceeding lemma to see that W has size 2 and $\dim(G/B) = 1$; the upper bound $\#W \leq 2$ follows from the fact that $W \hookrightarrow \mathrm{Aut}(T) \cong \{\pm 1\}$. We then argue that $G/B \cong \mathbb{P}^1$ using the algebraic fact: any smooth projective variety of dimension 1 which admits a nontrivial action by $\mathbb{G}_m$ is isomorphic to $\mathbb{P}^1$. We then consider the action of G on G/B. The kernel of this action is the intersection of the Borel subgroups, hence trivial. Thus $G \hookrightarrow \mathrm{Aut}(\mathbb{P}^1)$. On the other hand, by an exercise in algebraic geometry (making use of the "rational root theorem"), we have $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2(k)$, given by fractional linear transformations. Since G is non-solvable, we have $\dim(G) \geq 3$. Since $\mathrm{PGL}_2(k)$ is connected and 3-dimensional, we obtain the required surjection $G \to \mathrm{PGL}_2(k)$.

It remains to verify the refined conclusion when G is reductive, i.e., that $\ker(\rho)$ is diagonalizable. Let T be a maximal torus of G. It suffices to check that $\ker(\rho)$ is contained in every (in particular, some) maximal torus T of G. By the previous result, we have $\#\mathcal{B}^T = \#W(G,T) = 2$, using here that $\dim(G/B) = 1$. Write $\mathcal{B}^T = \{B^+, B^-\}$. Since $\ker(\rho)$ is the intersection of all Borel subgroups of G, it suffices to verifty that $B^+ \cap B^- = T$. Since $B^\pm = B_u^\pm T$, it is the same to show that $B_u^+ \cap B_u^- = 1$.

Note that Borel subgroups of $\mathrm{PGL}_2(k)$ are 2-dimensional and have 1-dimensional unipotent radicals.

On the other hand, recall that G has semisimple-rank 1, hence $B^\pm \neq G$. By the contrapositive of an earlier, we deduce that $B^\pm$ is not nilpotent. In particular, $B_u^\pm$ is nontrivial. It is also connected, being a factor in the semidirect product decomposition of B. Hence $\dim(B_u^\pm) \geq 1$.

Since $\ker(\rho)^0 \cap B_u^\pm$ is contained in $R(G)_u = 1$, we see that $\ker(\rho) \cap B_u^\pm$ is finite. Thus $\rho(B_u^\pm)$ has the same dimension ≥ 1 as $B_u^\pm$, and is a connected unipotent subgroup of $\mathrm{PGL}_2(k)$. But any such subgroup has dimension ≤ 1. We deduce that $\dim(B_u^\pm) = 1$. We've seen in characteristic zero (and remarked in finite characteristic) that this forces $B_u^\pm \cong \mathbb{G}_a$. The conjugation action of T on B_u^+ defines a character $T \to \mathbb{G}_m$. This character is nontrivial (because otherwise B^+ would be nilpotent), hence it has the dense orbit $\mathbb{G}_a - \{0\}$. The intersection $B_u^+ \cap B_u^-$ is T-invariant, so if that intersection is nontrivial, then it must be all of B_u^+, and so $B_u^+ = B_u^-$. But this forces the contradiction $B^+ = B_u^+ T = B_u^- T = B^-$. Thus $B_u^+ \cap B_u^- = 1$, as required. □

23. Reductive groups and root data

We followed the final chapter of Szamuely's notes.