Exercise Sheet 1

Exercise 1: The hyperbolic plane

Consider the hyperbolic n-space

$$\mathbb{H}^n = \left\{ p \in \mathbb{R}^{n+1} \colon b(p,p) = -1 \text{ and } p_{n+1} \geq 1 \right\}$$

defined by the bilinear form $b(p,q) = p_1 q_1 + \ldots + p_n q_n - p_{n+1} q_{n+1}$. The tangent space at a point $p \in \mathbb{H}^n$ is defined as

$$T_p \mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1} \colon \begin{array}{c} \text{There exists a smooth path } \gamma \colon (-1,1) \to \mathbb{H}^n \\ \text{such that } \gamma(0) = p \text{ and } \dot{\gamma}(0) = x \end{array} \right\}.$$

(1) Show that $T_p \mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1} \colon b(p,x) = 0 \right\}$.

(2) Show that $g_p = b|_{T_p \mathbb{H}^n} \colon T_p \mathbb{H}^n \times T_p \mathbb{H}^n \to \mathbb{R}$ is a positive definite symmetric bilinear form on $T_p \mathbb{H}^n$. This means that g_p is a scalar product and $(\mathbb{H}^n, g)$ is a Riemannian manifold.

Hint: Use (1) and the Cauchy-Schwarz-inequality on $\mathbb{R}^n$.

(3) Show that the map $s_p \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, q \mapsto -2p \cdot b(p,q) - q$ is a well defined geodesic symmetry, i.e. it is an involution, with an isolated fixed point p. This means that the hyperbolic plane $\mathbb{H}^n$ is a symmetric space.

Exercise 2: The symmetric space $\mathcal{P}^1(n)$

Show that $A \mapsto gA\,{}^t g$ defines a group action of $\mathrm{SL}(n,\mathbb{R}) \ni g$ on

$$\mathcal{P}^1(n) = \left\{ A \in M_{n \times n}(\mathbb{R}) \colon A = {}^t A, \quad \det A = 1, \quad A >> 0 \right\}.$$

Show that this action is transitive, i.e. $\forall A, B \in \mathcal{P}^1(n)\ \exists g \in \mathrm{SL}(n,\mathbb{R}) \colon gA\,{}^t g = B$. You may use the Linear-Algebra-fact that symmetric matrices are orthogonally diagonalizable, i.e. if $A = {}^t A$, then $\exists Q \in \mathrm{SO}(n,\mathbb{R})$ such that $QA\,{}^t Q$ is diagonal.

Exercise 3: Topological groups

A group G with a topology is a *topological group* if multiplication $m \colon G \times G \to G$ and inverse $\iota \colon G \to G$ are continuous maps. Let G be a topological group and $e \in G$ the identity.

(1) Show that $\forall g \in G$, the inner automorphism $\phi_g(h) = ghg^{-1}$ is a homeomorphism.

(2) Show that the connected component of the identity G° is a normal closed subgroup of G.

(3) Show that any open subgroup $H < G$ is also closed.

Hint: Cosets.

(4) Let $U \ni e$ be an open neighborhood of e. Let H be the subgroup generated by U, i.e.

$$H = \bigcup_{n \geq 1} \left(U \cup U^{-1}\right)^n .$$

Show that H is open.

(5) Show that connected groups are generated by any neighborhood of e.

Exercise 4: Lemma II.17

Recall that $\forall f \in C^\infty(M), X \in \operatorname{Vect}(M)$, we have $f \cdot X \in C^\infty(M)$ via $(f \cdot X)(p) = f(p)X(p)$. If c is a smooth curve, we denote by $\operatorname{Vect}(c^*(TM))$ the space of vector fields along c. Prove the following lemma.

Lemma II.17: Let M be a smooth manifold, ∇ a connection on M and $c\colon I \to M$ a smooth curve. Then there exists a unique linear map

$$\frac{\mathrm{D}}{dt}\colon \operatorname{Vect}\left(c^*(TM)\right) \to \operatorname{Vect}\left(c^*(TM)\right)$$

such that

(1) $\frac{\mathrm{D}}{dt}(f \cdot V) = f' \cdot V + f \cdot \frac{\mathrm{D}}{dt} V$ for all $V \in \operatorname{Vect}(c^*(TM)), f \in C^\infty(M)$

(2) $\left(\frac{\mathrm{D}}{dt} V\right)(t) = (\nabla_{\dot{c}(t)} Y)(c(t))$ for all $V \in \operatorname{Vect}(c^*(TM)), Y \in \operatorname{Vect}(M), t \in I$ with $V(t) = Y(c(t))$.

Hint: Work in local coordinates.

Exercise 5: Lemma II.20

Let now $\varphi\colon M \to M$ be a diffeomorphism. Recall that the pushforward $\varphi_* X$ is defined as $(\varphi_* X)(p) = \left(\mathrm{D}_{\varphi^{-1}(p)}\, \varphi\right) \left(X(\varphi^{-1}(p))\right)$ for $X \in \operatorname{Vect}(M), p \in M$. The goal is to prove the following lemma.

Lemma II.20: Let ∇ be the Levi-Civita connection of a Riemannian manifold (M, g) and $\varphi \in \operatorname{Is}(X)$. Then $\nabla_{\varphi_* X}(\varphi_* Y) = \varphi_*(\nabla_X Y)$.

(1) Show that $\mathrm{D}_X\, Y := \varphi_*^{-1} \left(\nabla_{\varphi_* X}(\varphi_* Y)\right)$ is a connection.

(2) Show that $\mathrm{D}_X\, Y - \mathrm{D}_Y\, X = [X, Y]$.

(3) Show that $X < Y, Z > = < \mathrm{D}_X\, Y, Z > + < Y, \mathrm{D}_X\, Z >$.

(4) Use that the Levi-Civita-connection is unique to conclude the lemma.