Exercise Sheet 2

Exercise 1: Prop II.22 (4)

Let $\gamma : \mathbb{R} \to M$ be a geodesic to a globally symmetric space M. Consider the transvection

$$\mathcal{T}_t^\gamma = s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)} \in \mathrm{Is}(M)^\circ$$

along γ. For $b \in \mathbb{R}$, $\eta(t) = \gamma(t+b)$ is also a geodesic and we can define the transvection $\mathcal{T}_t^\eta$ along η. Use the fact that $t \mapsto \mathcal{T}_t$ is a 1-parameter group (Prop II.22 (3)) to show that

$$\mathcal{T}_t^\gamma = \mathcal{T}_t^\eta.$$

Exercise 2: A symmetric space with non-compact K.

Let

$$A = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda > 0 \right\},$$
$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\},$$

then the Iwasawa-decomposition states that the map

$$\begin{aligned} \mathrm{SO}(2) \times A \times N &\to \mathrm{SL}(2,\mathbb{R}) \\ (k,a,n) &\mapsto kan \end{aligned}$$

is a diffeomorphism (but not a group-homomorphism).

(1) Explain why $\pi_1(\mathrm{SL}(2,\mathbb{R})) = \mathbb{Z}$.

(2) Check that $\sigma\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}), g \mapsto {}^t g^{-1}$ is an involution.

(3) By covering space theory we can lift σ to the universal cover $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Argue why $\tilde{\sigma}\colon \widetilde{\mathrm{SL}(2,\mathbb{R})} \to \widetilde{\mathrm{SL}(2,\mathbb{R})}$ is also an involution. You may use that the universal cover of a path-connected topological group is again a topological group.

(4) Prove that $\widetilde{\mathrm{SL}(2,\mathbb{R})}^{\tilde{\sigma}} = \widetilde{\mathrm{SO}(2)} \cong \mathbb{R}$.

Recall that for a closed subgroup $G < \mathrm{GL}(n,\mathbb{R})$, the adjoint representation is given by

$$\begin{aligned} \mathrm{Ad}_G\colon G &\to \mathrm{GL}(\mathfrak{g}) \\ g &\mapsto (X \mapsto gXg^{-1}) \end{aligned}$$

(5) Calculate the kernel of $\mathrm{Ad}_{\mathrm{SL}(2,\mathbb{R})}|_{\mathrm{SO}(2,\mathbb{R})}$ to see that

$$\mathrm{Ad}_{\mathrm{SL}(2,\mathbb{R})}(\mathrm{SO}(2,\mathbb{R})) = {}^{\mathrm{SO}(2,\mathbb{R})}/_{\pm 1}.$$

(6) Argue why $\mathrm{Ad}_{\widetilde{\mathrm{SL}(2,\mathbb{R})}}(\widetilde{\mathrm{SO}(2,\mathbb{R})}) = \mathrm{Ad}_{\mathrm{SL}(2,\mathbb{R})}(\mathrm{SO}(2,\mathbb{R}))$.

We set $G = \widetilde{\mathrm{SL}(2,\mathbb{R})}$ and $K = \widetilde{\mathrm{SO}(2,\mathbb{R})}$. Note that K is not compact but $\mathrm{Ad}_G(K)$ is. We therefore still get a symmetric space ${}^G/_K$.

Exercise 3: K-invariant scalar product

Let V be real vectorspace and $K < \mathrm{GL}(V)$ a compact subgroup. Prove that there exists a K-invariant scalar product on V.

Exercise 4: The center

(1) Let G be a connected topological group and $N \lhd G$ a normal subgroup which is discrete. Show that $N \subset Z(G)$ is contained in the center $Z(G)$ of G.

(2) Let (G, K) be a Riemannian symmetric pair and $Z(G)$ the center of G. Show that $\mathrm{Ad}_G \colon G \to \mathrm{GL}(\mathfrak{g})$ induces an isomorphism of Lie groups:

$$K/(K \cap Z(G)) \to \mathrm{Ad}_G(K) < \mathrm{GL}(\mathfrak{g}).$$