Exercise Sheet 3

Exercise 1: Exercise II.43: Properties of totally geodesic submanifolds

Assume $N \subset M$ is totally geodesic. Prove the following

(1) The inclusion $(N, d_N) \to (M, d_M)$ is locally distance-preserving.

(2) Every N-geodesic is an M-geodesic and every M-geodesic that is fully contained in N is an N-geodesic.

Exercise 2: Prop II.46: Parallel transport

Prove that the following are equivalent

(i) $N \subset M$ is totally geodesic.

(ii) The parallel transport for the Riemannian metric on M along curves contained in N preserves the tangent space distribution $\{T_pN : p \in N\}$.

Hint: Express $\frac{D}{dt}V(t) = 0$ *and* $\nabla_{\dot\gamma}\dot\gamma = 0$ *in local coordinates.*

Exercise 3: Derivative of the exponential map

We follow S. Sternberg's "Lie algebras" (see reference on website). We need to recall the notion of a differential form. Let V be a finite-dimensional vectorspace. Let $\bigwedge^k V$ denote the space of alternating k-forms on V. Given $\alpha \in \bigwedge^k V$, $\beta \in \bigwedge^l V$, the wedge product $\alpha \wedge \beta \in \bigwedge^{k+l} V$ is bilinear, associative and satisfies

$$\alpha \wedge \beta = (-1)^{k \cdot l} \beta \wedge \alpha.$$

Now let G be a Lie group and $\mathfrak{g}$ its Lie algebra. A $\mathfrak{g}$-valued differential k-form ω on G, associates (in a smooth way) to every point $g \in G$ a k-form

$$\omega_g \in \bigwedge^k \{T_gG \to \mathfrak{g} \colon \text{linear }\}$$

The set of all $\mathfrak{g}$-valued differential k-forms is denoted by $\Omega^k(G, \mathfrak{g})$. The exterior derivative $d \colon \Omega^k(G, \mathfrak{g}) \to \Omega^{k+1}(G, \mathfrak{g})$ is defined by the following three properties.

(i) For a differential 0-form $f \colon G \to \mathfrak{g}$, the exterior derivative $df \in \Omega^1(G, \mathfrak{g})$ is given by the total derivative $df_g = \mathrm{D}_g f \colon T_gG \to \mathfrak{g}$.

(ii) $d \circ d = 0$.

(iii) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k(\alpha \wedge d\beta)$ for $\alpha \in \Omega^k(G, \mathfrak{g})$.

The Maurer-Cartan-form $\theta \in \Omega^1(G, \mathfrak{g})$ is defined by

$$\theta_A = \mathrm{D}_A L_{A^{-1}} \colon T_AG \to T_{\mathrm{Id}}G = \mathfrak{g}$$

where L is lefttranslation and $A \in G$.

(1) Now let $G < \mathrm{GL}(n, \mathbb{R})$. Interpret A as the identity function $A\colon G \mapsto M_{n\times n}, a \mapsto a$. Then $dA_a = \mathrm{D}_a A\colon T_aG \to M_{n\times n}, X \mapsto X$ is also the identity. Prove that $\theta = A^{-1} \cdot dA$.

Hint: Choose $\gamma\colon \mathbb{R} \to G$ with $\gamma(0) = a, \dot\gamma(0) \in T_aG$.

(2) Prove $d(A^{-1}) = -A^{-1} \cdot dA \cdot A^{-1}$.

(3) Prove $d\theta = -\theta \wedge \theta$ (Maurer-Cartan-equation).

Hint: Use associativity of $\wedge$-product:

$$\omega \cdot X \wedge \eta = \omega \wedge X \wedge \eta = \omega \wedge X \cdot \eta$$

for $X \in \mathfrak{g}$ and $\omega, \eta \in \Omega^k(G, \mathfrak{g})$.

(4) Let $g\colon s\,\mathbb{R} \oplus t\,\mathbb{R} = \mathbb{R}^2 \to G$. The pullback $g^*\theta \in \Omega^1(\mathbb{R}^2, \mathfrak{g})$, defined by $g^*\theta_{(s,t)} = \theta_{g(s,t)}\, \mathrm{D}_{(s,t)}\, g$, is a differential 1-form and can be written as

$$g^*\theta = \alpha \cdot ds + \beta \cdot dt$$

for some $\alpha, \beta\colon \mathbb{R}^2 \to \mathfrak{g}$. Show that

$$g^*\theta_{(s,t)} \cdot \begin{pmatrix}1\\0\end{pmatrix} = \alpha(s,t)$$
$$g^*\theta_{(s,t)} \cdot \begin{pmatrix}0\\1\end{pmatrix} = \beta(s,t)$$

and use it to calculate

$$\alpha(s,t) = g(s,t)^{-1} \cdot \frac{\partial g}{\partial s}(s,t)$$
$$\beta(s,t) = g(s,t)^{-1} \cdot \frac{\partial g}{\partial t}(s,t).$$

Hint: ds is the derivative of $s\colon \mathbb{R}^2 \to \mathbb{R}$.

(5) Prove $(\alpha ds + \beta dt) \wedge (\alpha ds + \beta dt) = [\alpha, \beta] ds \wedge dt$.

(6) Prove $d\alpha = \frac{\partial \alpha}{\partial s} ds + \frac{\partial \alpha}{\partial t} dt$.

(7) The pullback commutes with d and $\wedge$, so

$$d(g^*\theta) + g^*\theta \wedge g^*\theta = 0$$

follows from the Maurer-Cartan equation (3). Prove

$$\frac{\partial \beta}{\partial s} - \frac{\partial \alpha}{\partial t} + [\alpha, \beta] = 0.$$

(8) Let $C\colon \mathbb{R} \to \mathfrak{g}$ be a curve. Define $g(s,t) = \exp(sC(t))$. Calculate $\alpha(s,t)$ in this case. Then prove

$$\frac{\partial \beta}{\partial s} - C'(t) + [C(t), \beta] = 0 \qquad \text{and}$$
$$\beta(0,t) = 0.$$

(9) We now fix t and set $C = C(t), C' = C'(t)$. We get the differential equation

$$\begin{aligned}\frac{d\beta}{ds} &= -(\operatorname{ad} C)\beta + C',\\ \beta(0) &= 0.\end{aligned}$$

Use the Taylor expansion to get the solution

$$\beta(s,t) = \sum_{n=1}^{\infty} \frac{s^n(-\operatorname{ad} C(t))^{n-1}}{n!} \cdot C'(t).$$

Combine this with (4) and put $s = 1$ to get

$$\exp(-C(t)) \cdot \frac{d}{dt}\exp(C(t)) = \left(\sum_{n=0}^{\infty} \frac{(-\operatorname{ad}(C(t)))^n}{(n+1)!}\right) C'(t).$$

(10) Denote $X = C(0)$ and $\eta = C'(0)$. Show that

$$\exp(-C(t)) \cdot \frac{d}{dt}\exp(C(t))\bigg|_{t=0} = \theta_{\exp(X)} D_X \exp \eta$$

and conclude the formula for the derivative of the exponential function (Thm II.40)

$$\mathrm{D}_X \exp = \left(\mathrm{D}_e\, L_{\exp(X)}\right)\left(\sum_{n=0}^{\infty} \frac{(-\operatorname{ad}(X))^n}{(n+1)!}\right).$$