Exercise Sheet 4

Exercise 1: Compact symmetric spaces

Let K be a compact connected Lie group with $\dim(K) \geq 1$. Define $G = K \times K$ and $\sigma(g,h) = (h,g)$ for $(g,h) \in G$.

(1) Show that (G, G^σ) is a Riemannian symmetric pair with involution σ.

(2) Consider the action $(g,h).k = gkh^{-1}$ of $(g,h) \in G$ on $k \in K$. Show that

$$G/G^\sigma \to K$$

is a homeomorphism.

Recall that by the Hopf-Rinow-theorem, the following are equivalent for a Riemannian manifold M:

- The closed bounded subsets of M are compact.
- M is complete as a metric space.
- M is geodesically complete, i.e. $\forall p \in M, \mathrm{Exp}_p \colon T_pM \to M$ is defined on the entire tangent space T_pM.

Moreover, if M satisfies the above, then any two points $p, q \in M$ can be joined by a (minimal) geodesic.

(3) Use the Hopf-Rinow theorem to show that the Lie group exponential is surjective.

Exercise 2: Theorem III.9: Classification of effective OSP

Let $(\mathfrak{g}, \theta)$ be an effective orthogonal symmetric Lie-algebra. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ $(= \mathfrak{k} \oplus \mathfrak{p})$. We decomposed $\mathfrak{e} = \mathfrak{e}_0 \oplus \mathfrak{e}_+ \oplus \mathfrak{e}_-$ and defined $\mathfrak{u}_+ = [\mathfrak{e}_+, \mathfrak{e}_+]$ and $\mathfrak{u}_- = [\mathfrak{e}_-, \mathfrak{e}_-]$. $\mathfrak{u}_0$ is defined to be the orthogonal complement of $\mathfrak{u}_+ \oplus \mathfrak{u}_-$ in $\mathfrak{u}$.

(1) Prove that $\mathfrak{u}_- \oplus \mathfrak{e}_-$ and $\mathfrak{u}_+ \oplus \mathfrak{e}_+$ are ideals in $\mathfrak{g}$.

(2) Prove that $\mathfrak{u}_0 \oplus \mathfrak{e}_0$, $\mathfrak{u}_- \oplus \mathfrak{e}_-$ and $\mathfrak{u}_+ \oplus \mathfrak{e}_+$ are θ-stable and pairwise orthogonal with respect to $B_\mathfrak{g}$.

(3) Find an OSL $(\mathfrak{g}, \theta)$, such that $\mathfrak{e}_0 = 0$, but $\mathfrak{u}_0 \neq 0$.

(4) Let $\mathfrak{n} \lhd \mathfrak{g}$ be an ideal of a Lie-algebra $\mathfrak{g}$. Prove that $B_\mathfrak{n} = B_\mathfrak{g}|_{\mathfrak{n} \times \mathfrak{n}}$.

(5) Find an example of a subalgebra $\mathfrak{n} \subset \mathfrak{g}$, such that $B_\mathfrak{n} \neq B_\mathfrak{g}|_{\mathfrak{n} \times \mathfrak{n}}$.

(6) Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ a direct sum of two ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Further let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be subalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Show that $\mathfrak{k}_1 + \mathfrak{k}_2$ is compactly embedded in $\mathfrak{g}$ if and only if $\mathfrak{k}_1$ and $\mathfrak{k}_2$ is compactly embedded in $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

This implies that $\mathfrak{u}_0, \mathfrak{u}_-, \mathfrak{u}_+$ are compactly embedded in $\mathfrak{g}_0, \mathfrak{g}_-$ and $\mathfrak{g}_+$.

Hint: For connected G and $K < G$, there is an isomorphism

$$K/(K \cap Z(G)) \cong \mathrm{Ad}_G(K)$$

(compare Sheet 2, exercise 4(2)). Use $\mathrm{Lie}(\mathrm{Ad}_G(K)) = \mathrm{ad}_{\mathrm{Lie}(G)}(\mathrm{Lie}(K))$.

Exercise 3: Theorem III.19: Classification of s.c. RSS

(1) Let $H, N \lhd G$ be two normal subgroups. Show that $[N, H] \subset N \cap H$.

(2) Let $H, N < G$ be connected subgroups. Show that $[N, H]$ is a connected subgroup of G.

Let M be a simply connected Riemannian symmetric space. Then $\mathfrak{g} = \mathrm{Lie}(\mathrm{Is}(M)^\circ) = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$. We get corresponding Lie-subgroups G_0, G_+, G_- and their universal covers $\tilde{G}_0, \tilde{G}_+, \tilde{G}_-$. Let K_0, K_+, K_- be the Lie-subgroups associated to $\mathfrak{k}_0, \mathfrak{k}_+, \mathfrak{k}_-$, which come from the Cartan-decomposition of $\mathfrak{g}_0, \mathfrak{g}_+, \mathfrak{g}_-$.

(3) Show that $(\tilde{G}_0, K_0), (\tilde{G}_+, K_+)$ and $(\tilde{G}_-, K_-)$ are Riemannian symmetric pairs.