Exercise Sheet 5

Exercise 1: Theorem III.22: Decomposition into irreducible parts

Let $(\mathfrak{g}, \theta)$ be a reduced OSL with Cartan decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$. We can write

$$\mathfrak{e} = \mathfrak{e}_0 \oplus \bigoplus_{j=1}^{a} \mathfrak{p}_j,$$

where each $\mathfrak{p}_j$ is U-invariant and irreducible. We define $\mathfrak{g}_j = [\mathfrak{p}_j, \mathfrak{p}_j] + \mathfrak{p}_j$.

(1) Show that $\mathfrak{g}_j$ is a θ-stable ideal in $\mathfrak{g}$ for $1 \leq j \leq a$.

(2) From (1) and Sheet 4, exercise 2(4) it follows that $B_{\mathfrak{g}_j} = B_{\mathfrak{g}}|_{\mathfrak{g}_j \times \mathfrak{g}_j}$. Show that $B_{\mathfrak{g}_j}$ is non-degenerate, i.e. $\mathfrak{g}_j$ is semi-simple.

(3) Let

$$\mathfrak{m} = \bigoplus_{j=1}^{a} \mathfrak{g}_j \,.$$

Prove that the centralizer

$$\mathfrak{g}_0 := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{m}) = \{X \in \mathfrak{g} \colon [X, m] = 0 \ \forall m \in \mathfrak{m}\}$$

is a θ-stable ideal in $\mathfrak{g}$.

(4) Show that this gives rise to an orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{m}$.

(5) Show that $(\mathfrak{g}_0, \theta|_{\mathfrak{g}_0})$ is a reduced OSL of Euclidean type (or 0).

(6) Show that the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{j=1}^{a} \mathfrak{g}_j$ is unique up to reordering.

Hint: Consider two decompositions and the projection $\pi \colon \mathfrak{g} \to \mathfrak{g}_0$. Show that the projection of a semi-simple ideal is a semi-simple ideal. Use that the Killing form of an ideal is the restriction of $B_{\mathfrak{g}}$ to the ideal.

Exercise 2: Lemma IV.10: Automorphism group

Let $\mathfrak{g}$ be a real Lie-algebra. Prove that $\mathrm{Aut}(\mathfrak{g})$ is a Lie-subgroup of $\mathrm{GL}(\mathfrak{g})$ with Lie-algebra is $\mathrm{Der}(\mathfrak{g})$.

Hint: The Lie algebra of a Lie-subgroup $H < G$ is

$$\mathrm{Lie}(H) = \{X \in \mathrm{Lie}(G) \colon \forall t \in \mathbb{R} \colon \exp(tX) \in H\}\,.$$

Exercise 3: Thm IV.15: Decomposition of the automorphism-group

Now let $(\mathfrak{g}, \theta)$ be a semisimple reduced OSL, which we write as a product $\mathfrak{g} = \bigoplus_{i=1}^{a} \mathfrak{g}_i$ of irreducible OSLs $(\mathfrak{g}_i, \theta_i)$, note that we do not have a Euclidean component since $\mathfrak{g}$ is semisimple. Let $G = \mathrm{Aut}(\mathfrak{g})^\circ$ and $G_i = \mathrm{Aut}(\mathfrak{g}_i)^\circ$. Let $\sigma \colon G \to G, \alpha \mapsto \theta \alpha \theta^{-1}$ and σ_i analogously. Prove that there is a Lie-group-isomorphism $\mathrm{Aut}(\mathfrak{g})^\circ \to \Pi_{i=1}^{a} (\mathrm{Aut}(\mathfrak{g}_i))^\circ$ which sends the subgroup G^σ to $\Pi_{i=1}^{a} G_i^{\sigma_i}$

Hint: Use uniqueness of the decomposition of OSLs.

Exercise 4: Thm IV.13: Properties of non-compact type

Show that the map $\varphi\colon \mathfrak{p} \times K \to G, (X,k) \mapsto \exp(X)k$ is regular (the derivative has full rank).

Hint: Recall that

$$\mathrm{D}_X \exp = \mathrm{D}_e\, L_{\exp(X)} \cdot \sum_{n=0}^{\infty} \frac{(-\operatorname{ad}(X))^n}{(n+1)!}$$

and decompose $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$*, so that you can use*

$$\det\left(\sum_{n=0}^{\infty} \frac{(\operatorname{ad}(X)^2|_{\mathfrak{p}})^n}{(2n+1)!}\right) \geq 1.$$