Solution Sheet 1

Exercise 1: The hyperbolic plane

Consider the hyperbolic n-space

$$\mathbb{H}^n = \left\{p \in \mathbb{R}^{n+1} \colon b(p,p) = -1 \text{ and } p_{n+1} \geq 1\right\}$$

defined by the bilinear form $b(p,q) = p_1 q_1 + \ldots + p_n q_n - p_{n+1} q_{n+1}$. The tangent space at a point $p \in \mathbb{H}^n$ is defined as

$$T_p \mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1} \colon \begin{array}{c} \text{There exists a smooth path } \gamma\colon (-1,1) \to \mathbb{H}^n \\ \text{such that } \gamma(0) = p \text{ and } \dot\gamma(0) = x \end{array} \right\}.$$

(1) Show that $T_p \mathbb{H}^n = \left\{x \in \mathbb{R}^{n+1} \colon b(p,x) = 0\right\}$.

Solution:

Let $x \in T_p \mathbb{H}^n$. Let $\gamma\colon (-1,1) \to \mathbb{H}^n$ be a smooth path such that $\gamma(0) = p$ and $\dot\gamma(0) = x$. For every $t \in (-1,1)$, $b(\gamma(t), \gamma(t)) = -1$, since γ takes values in $\mathbb{H}^n$. We write $\gamma(t) = (\gamma_1(t), \cdots, \gamma_{n+1}(t))$. Taking derivatives results in

$$0 = \frac{d}{dt} b(\gamma(t),\gamma(t)) = \frac{d}{dt}\left(\sum_{i=1}^n \gamma_i(t)^2 - \gamma_{n+1}^2\right) = \sum_{i=1}^n 2\gamma_i(t)\dot\gamma_i(t) - 2\gamma_{n+1}(t)\dot\gamma_{n+1}(t)$$

and at $t = 0$ this is

$$0 = \sum_{i=1}^n \gamma_i(0)\dot\gamma_i(0) - \gamma_{n+1}(0)\dot\gamma_{n+1}(0) = \sum_{i=1}^n p_i x_i - p_{n+1} \cdot x_{n+1} = b(p,x).$$

We have shown that $T_p \mathbb{H}^n \subset \left\{x \in \mathbb{R}^{n+1} \colon b(p,x) = 0\right\}$ but since $\dim T_p \mathbb{H}^n = n$ we have equality.

(2) Show that $g_p = b|_{T_p \mathbb{H}^n} \colon T_p \mathbb{H}^n \times T_p \mathbb{H}^n \to \mathbb{R}$ is a positive definite symmetric bilinear form on $T_p \mathbb{H}^n$. This means that g_p is a scalar product and $(\mathbb{H}^n, g)$ is a Riemannian manifold.

Hint: Use (1) and the Cauchy-Schwarz-inequality on $\mathbb{R}^n$.

Solution:

Bilinearity and symmetry $b(x,y) = b(y,x)$ follow directly. To show positive definiteness, we use the definition of $\mathbb{H}^n$ and (1) to write

$$p = \left(\vec{p}, \sqrt{|\vec{p}|^2 + 1}\right) \in \mathbb{H}^n \subset \mathbb{R}^n \times \mathbb{R}$$

$$x = \left(\vec{x}, \frac{< \vec{p}, \vec{x} >}{\sqrt{|\vec{p}|^2 + 1}}\right) \in T_p \mathbb{H}^n \subset \mathbb{R}^n \times \mathbb{R}$$

where $< \cdot, \cdot >$ is the standard scalar product in $\mathbb{R}^n$. To show positive definiteness it remains to prove that for all $x \in T_p \mathbb{H}^n$

$$b(x,x) \geq 0.$$

Indeed, by the Cauchy-Schwarz-inequality

$$< p, x >^2 \leq |\overrightarrow{p}|^2 |\overrightarrow{x}|^2 \leq |\overrightarrow{p}|^2 |\overrightarrow{x}|^2 + |\overrightarrow{x}|^2 = (|\overrightarrow{p}|^2 + 1)|\overrightarrow{x}|^2$$

and thus

$$|\overrightarrow{x}|^2 \geq \frac{< p, x >^2}{|\overrightarrow{p}|^2} + 1$$

and

$$b(x, x) = |\overrightarrow{x}|^2 - \frac{< p, x >^2}{|\overrightarrow{p}|^2 + 1} \geq 0.$$

(3) Show that the map $s_p \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, q \mapsto -2p \cdot b(p, q) - q$ is a well defined geodesic symmetry of $\mathbb{H}^n$, i.e. it is an involution, with an isolated fixed point p. This means that the hyperbolic plane $\mathbb{H}^n$ is a symmetric space.

Solution:

To see that s_p is well-defined we write

$$p = \begin{pmatrix} \overrightarrow{p} \\ \sqrt{|\overrightarrow{p}|^2 + 1} \end{pmatrix}, \quad q = \begin{pmatrix} \overrightarrow{q} \\ \sqrt{|\overrightarrow{q}|^2 + 1} \end{pmatrix} \in \mathbb{H}^n \subset \mathbb{R}^n \times \mathbb{R}\,.$$

We have

$$s_p(q) = -2pb(p, q) - q = \begin{pmatrix} -2\overrightarrow{p}b(p, q) - \overrightarrow{q} \\ -2\sqrt{|\overrightarrow{p}|^2 + 1}b(p, q) - \sqrt{|\overrightarrow{q}|^2 + 1} \end{pmatrix}$$

where

$$b(p, q) =< \overrightarrow{p}, \overrightarrow{q} > - \sqrt{|\overrightarrow{p}|^2 + 1}\sqrt{|\overrightarrow{q}|^2 + 1}.$$

The calculation

$$\begin{aligned} b(s_p(q), s_p(q)) &= 4|\overrightarrow{p}|^2 b(p, q)^2 + 4 < \overrightarrow{p}, \overrightarrow{q} > b(p, q) + |\overrightarrow{q}|^2 \\ &\quad - \left[4(|\overrightarrow{p}|^2 + 1)b(p, q)^2 + 4\sqrt{|\overrightarrow{p}|^2 + 1}\sqrt{|\overrightarrow{q}|^2 + 1}b(p, q) + |\overrightarrow{q}|^2 + 1 \right] \\ &= 4 < \overrightarrow{p}, \overrightarrow{q} > b(p, q) - 4b(p, q)^2 - 4\sqrt{|\overrightarrow{p}|^2 + 1}\sqrt{|\overrightarrow{q}|^2 + 1}b(p, q) - 1 \\ &= 4b(p, q)b(p, q) - 4b(p, q)^2 - 1 = -1 \end{aligned}$$

shows that $s_p(q) \in \mathbb{H}^2$.

Note that $s_p(p) = -2p(-1) - p = p$ is a fixed point.

Next we show that s_p is an isometry. We need to look at the differential

$$\mathrm{D}_p\, s_p \colon T_pM \to T_{s_p(p)}M = T_pM.$$

of $s_p \colon q \mapsto -2pb(p, q) - q$. If we write the points $q, p \in \mathbb{H}^n \subset \mathbb{R}^{n+1}$ in the standard basis $\{e_i\}_i$, we get the partial derivatives

$$\frac{\partial}{\partial x_i} b(p, \cdot) = \begin{cases} p_i & \text{if } i \leq n \\ -p_{n+1} & \text{if } i = n + 1 \end{cases}$$

$$\frac{\partial}{\partial x_i} s_p = \begin{cases} -2p \cdot p_i - e_i & \text{if } i \leq n \\ 2p \cdot p_{n+1} - e_{n+1} & \text{if } i = n + 1 \end{cases}$$

and thus for $v \in T_pM$ we have

$$\begin{aligned}
(\mathrm{D}_p\, s_p)v &= \begin{pmatrix} -2p_1^2-1 & -2p_1p_2 & \cdots & 2p_1p_{n+1} \\ -2p_2p_1 & -2p_2^2-1 & \cdots & 2p_2p_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ -2p_{n+1}p_1 & -2p_{n+1}p_2 & \cdots & 2p_{n+1}^2-1 \end{pmatrix} v \\
&= \begin{pmatrix} -2p_1^2v_1 - 2p_1p_2v_2 - \cdots + 2p_1p_{n+1}v_{n+1} \\ -2p_2p_1v_1 - 2p_2^2v_2 - \cdots + 2p_2p_{n+1}v_{n+1} \\ \vdots \\ -2p_{n+1}p_1v_1 - 2p_{n+1}p_2v_2 - \cdots + 2p_{n+1}^2v_{n+1} \end{pmatrix} - v \\
&= \begin{pmatrix} -2b(p,v)p_1 \\ -2b(p,v)p_2 \\ \vdots \\ -2b(p,v)p_n \end{pmatrix} - v = -v
\end{aligned}$$

where we used that $b(p, v) = 0$ from part (1). By bilinearity from (2)

$$g_{s_p(p)}((D_ps_p)v, (D_ps_p)w) = g_p(-v, -w) = g_p(v, w),$$

so s_p is an isometry.

We need to show that s_p is a symmetry. By lemma II.6 of the lecture, $D_ps_p = -\operatorname{Id}_{T_p\mathbb{H}^n}$ is equivalent to $s_p \circ s_p = \operatorname{Id}_{\mathbb{H}^n}$. Alternatively the calculation

$$\begin{aligned}
s_p \circ s_p(q) &= s_p(-2pb(p,q) - q) \\
&= -2pb(p, -2pb(p,q) - q) - (-2pb(p,q) - q) \\
&= 4pb(p,q)b(p,p) + 2pb(p,q) + 2pb(p,q) + q = q
\end{aligned}$$

shows the same. That p is an isolated fixed point of s_p can be seen by the following argument. Let $q \in \mathbb{H}^n$ be a fixed point $s_p(q) = q$, then $-2pb(p,q) - q = q$, so $q = -b(p,q)p$, in particular $q = \lambda p$ is a scaled version of p for $\lambda = -b(p,q)$. But since $p, q \in \mathbb{H}^n$, $-1 = b(q,q) = b(\lambda p, \lambda p) = \lambda^2 b(p,p) = -1$, so $\lambda = \pm 1$. The $\lambda = -1$ solution corresponds to $q_{n+1} < 0$ which is excluded since $\mathbb{H}^n$ is only the upper hyperboloid. We showed that $q = p$ is the only fixed point of s_p, in particular it is an isolated fixed point.

This concludes the proof, that $\mathbb{H}^n$ is a symmetric space.

Exercise 2: The symmetric space $\mathcal{P}^1(n)$

Show that $A \mapsto gA\,{}^tg$ defines a group action of $\mathrm{SL}(n,\mathbb{R}) \ni g$ on

$$\mathcal{P}^1(n) = \left\{A \in M_{n\times n}(\mathbb{R})\colon A = {}^tA, \quad \det A = 1, \quad A >> 0\right\}.$$

Show that this action is transitive, i.e. $\forall A, B \in \mathcal{P}^1(n)\ \exists g \in \mathrm{SL}(n,\mathbb{R})\colon gA\,{}^tg = B$. You may use the Linear-Algebra-fact that symmetric matrices are orthogonally diagonalizable, i.e. if $A = {}^tA$, then $\exists Q \in \mathrm{SO}(n,\mathbb{R})$ such that $QA\,{}^tQ$ is diagonal.
Solution:

We write the group action as $g.A = gA\,{}^tg$. We first need to show that the action is well defined.

Symmetry: ${}^t(g.A) = {}^t(gA\,{}^tg) = g\,{}^tA\,{}^tg = gA\,{}^tg = g.A$.

Determinant: $\det(g.A) = \det g \det A \det {}^t g = \det A = 1$.

Positive definiteness: Let $x \in \mathbb{R}^n \setminus 0$. ${}^t x g.A x = {}^t x g A\, {}^t g x = {}^t({}^t g x) A\, {}^t g x > 0$, since ${}^t g x \in \mathbb{R}^n \setminus 0$. Next, we check the two axioms of a group action.

Identity: $\mathrm{Id}_{\mathrm{SL}(n,\mathbb{R})} .A = \mathrm{Id}\, A\, \mathrm{Id} = A$.

Compatibility: $(gh).A = ghA\, {}^t(gh) = ghA\, {}^t h\, {}^t g = g(h.A)\, {}^t g = g.(h.A)$.

It remains to show that the action is transitive. Let $A, B \in \mathcal{P}^1(n)$. We can use Linear Algebra to get $Q, R \in \mathrm{SO}(n) < \mathrm{SL}(n, \mathbb{R})$ such that $Q.A$ and $R.B$ are diagonal, have determinant 1 and are positive definite (by the well-definedness of the group action). Positive definiteness implies that all entries are non-negative. Then the matrix $\Lambda = (Q.A) \cdot (R.B)^{-1}$ is also diagonal, has determinant 1 and positive elements on the diagonal. We can therefore take the component wise root $\sqrt{\Lambda}$ of Λ. Define $g = Q^{-1}\sqrt{\Lambda} R \in \mathrm{SL}(n, \mathbb{R})$ and use the fact that $R.B$ commutes with $\sqrt{\Lambda}$ since they are diagonal to see that

$$\begin{aligned} g.B &= Q^{-1}.\sqrt{\Lambda}.R.B = Q^{-1}.\sqrt{\Lambda}(R.B)\, {}^t\sqrt{\Lambda} = Q^{-1}.\left(\sqrt{\Lambda}\, {}^t\sqrt{\Lambda} \cdot R.B\right) \\ &= Q^{-1}.(\Lambda \cdot R.B) = Q^{-1}.((Q.A)(R.B)^{-1}(R.B)) = Q^{-1}.Q.A = A. \end{aligned}$$

this shows that from any point $B \in \mathcal{P}$ you can go to any point $A \in \mathcal{P}$ by the action of $\mathrm{SL}(n, \mathbb{R})$, i.e. the action is transitive.

Exercise 3: Topological groups

A group G with a topology is a *topological group* if multiplication $m \colon G \times G \to G$ and inverse $\iota \colon G \to G$ are continuous maps. Let G be a topological group and $e \in G$ the identity.

(1) Show that $\forall g \in G$, the inner automorphism $\phi_g(h) = ghg^{-1}$ is a homeomorphism.

Solution:

m and ι are continuous, so also the composition $\phi_g \colon h \mapsto m(m(g, h), \iota(g))$ for any $g \in G$. Note that $\phi_g^{-1} = \phi_{g^{-1}}$, so the inverse is also continuous, i.e. ϕ_g is homeomorph.

(2) Show that the connected component of the identity G° is a normal closed subgroup of G.

Solution:

The image under a continuous map of a connected set is connected. Let $g, h \in G^\circ$. First consider the continuous map $a \mapsto m(g, a)$. Since h is in the same connected component as e, also $m(g, h) = gh$ is in the same connected component as $m(g, e) = g$, which is G°. Since $\iota(e) = e$, also $\iota(g) = g^{-1}$ is in the same connected component as g. Therefore G° is a subgroup of G.

The image $\phi_g(G^\circ)$ is connected and contains e, therefore $gG^\circ g^{-1} \subset G^\circ$, i.e. G° is normal.

Connected components are always open and closed.

(3) Show that any open subgroup $H < G$ is also closed.

Hint: Cosets.

Solution:
The coset gH is also open, since it is the preimage of H under the continuous map $h \mapsto m(g^{-1}, h)$. The complement of H is a union of open cosets, therefore H is closed.

(4) Let $U \ni e$ be an open neighborhood of e. Let H be the subgroup generated by U, i.e.

$$H = \bigcup_{n \geq 1} \left(U \cup U^{-1}\right)^n .$$

Show that H is open.

Solution:
If U is open, then also $\iota(U) = U^{-1}$ open and $U \cup U^{-1}$ open. For any $g \in G$, gU and gU^{-1} are open, since they are preimages of the continuous map $h \mapsto g^{-1}h$. Using $gU \cup gU^{-1} = g(U \cup U^{-1})$ we get that

$$(U \cup U^{-1})^n = \bigcup_{g \in U} g(U \cup U^{-1})^{n-1}$$

is open for any $n \geq 2$ by induction. Thus H is a union of open sets and therefore open.

(5) Show that G° is generated by any neighborhood of e.

Solution:
Any neighborhood of e contains an open neighborhood U. By the construction of (4), this generates an open subgroup H. By (3) H is also closed. The only clopen sets in a connected component are the empty set and the component itself. Since $e \in H$ and G° connected, $H = G^\circ$.

Exercise 4: Lemma II.17

Recall that $\forall f \in C^\infty(M), X \in \mathrm{Vect}(M)$, we have $f \cdot X \in C^\infty(M)$ via $(f \cdot X)(p) = f(p)X(p)$. If c is a smooth curve, we denote by $\mathrm{Vect}(c^*(TM))$ the space of vector fields along c. Prove the following lemma.

Lemma II.17: Let M be a smooth manifold, ∇ a connection on M and $c\colon I \to M$ a smooth curve. Then there exists a unique linear map

$$\frac{\mathrm{D}}{dt}\colon \mathrm{Vect}\left(c^*(TM)\right) \to \mathrm{Vect}\left(c^*(TM)\right)$$

such that

(1) $\frac{\mathrm{D}}{dt}(f \cdot V) = f' \cdot V + f \cdot \frac{\mathrm{D}}{dt} V$ for all $V \in \mathrm{Vect}(c^*(TM)), f \in C^\infty(M)$

(2) $\left(\frac{\mathrm{D}}{dt} V\right)(t) = (\nabla_{\dot{c}(t)} Y)(c(t))$ for all $V \in \mathrm{Vect}(c^*(TM)), Y \in \mathrm{Vect}(M), t \in I$ with $V(t) = Y(c(t))$.

Hint: Work in local coordinates.
Solution:
Let M be a n-dimensional manifold. Since c is a smooth immersion, for every point $c(t_0)$ we can find $\varepsilon > 0$, a neighborhood $U \ni c(t_0)$ and a chart $\psi : U \to \mathbb{R}^n$

such that $\psi(c(t_0)) = 0$ and for $t \in (-\varepsilon, \varepsilon)$, $\psi(U \cap c(t)) = t \times \{0\}^{n-1} \subset \mathbb{R}^n$. This means that we can assume without loss of generality that $M \subset \mathbb{R}^n$ and $c(t) = t$ for $t \ni I \subset \mathbb{R} \times \{0\}^{n-1}$.

Let $V : I \to TM$ be a vector field along c and $Y, Y' \in \mathrm{Vect}(M)$ with $Y(c(t)) = V(t) = Y'(c(t))$. We can write $Y = \sum_{j=1}^n y_j e_j$ and $Y' = \sum_{j=1}^n y_j' e_j$ for functions $y_j, y_j' : I \to \mathbb{R}$. Let $X \in \mathrm{Vect}(M)$ be defined by $X(p) = e_1 = \frac{\partial}{\partial x_1}$, in particular $X(c(t)) = \dot{c}(t)$ for $t \in I$.

Let us define $\left(\frac{\mathrm{D}}{dt} V\right)(t) = (\nabla_X Y)(c(t))$. We have to show that this is well defined (independent of Y). Note that

$$\frac{\partial}{\partial x_1} Y(c(t_0)) = \frac{\partial}{\partial x_1} Y'(c(t_0)) = \frac{d}{dt} V(t_0). \tag{$\clubsuit$}$$

We have

$$\begin{aligned}
\left(\frac{\mathrm{D}}{dt} V\right)(t) &= \nabla_X Y(c(t_0)) \\
&= \nabla_{e_1} \left(\sum_{j=1}^n y_j e_j \right) (c(t_0)) \\
&= \left[\sum_{j=1}^n y_j \cdot \nabla_{e_1} e_j + (e_1 y_j) \cdot e_j \right] (c(t_0)) && \text{(Rule (3) for connections)} \\
&= \left[\sum_{j,k=1}^n y_j \cdot \Gamma_{1j}^k \cdot e_k + \sum_{j=1}^n \frac{\partial}{\partial x_1} y_j \cdot e_j \right] (c(t_0)) && \text{(for Christoffel symbols } \Gamma_{ij}^k : M \to \mathbb{R}) \\
&= \left[\sum_{j,k=1}^n y_j \cdot \Gamma_{1j}^k \cdot e_k \right] (c(t_0)) + \frac{\partial}{\partial x_1} Y(c(t_0)) \\
&= \left[\sum_{j,k=1}^n y_j \cdot \Gamma_{1j}^k \cdot e_k \right] (c(t_0)) + \frac{d}{dt} V(t_0), && \text{(Equation } (\clubsuit))
\end{aligned}$$

which is an expression which does not depend on Y. This shows that a map $\frac{\mathrm{D}}{dt} : \mathrm{Vect}(c_* TM) \to \mathrm{Vect}(c_* TM)$ that satisfies (2) exists and is unique.

We have to show (1). Let $f : I \to \mathbb{R}$. We can extend it to $\tilde{f} \in \mathrm{C}^\infty(M)$ with $\tilde{f}(t, 0^{n-1}) = f(t)$ for $t \in I$. then

$$\begin{aligned}
\frac{\mathrm{D}}{dt}(f \cdot V)(t_0) &= \nabla_X (\tilde{f} \cdot Y)(c(t_0)) \\
&= \left[\tilde{f} \cdot \nabla_X Y + (X \tilde{f}) \cdot Y \right] (c(t_0)) \\
&= \left(f \cdot \frac{\mathrm{D}}{dt} V \right)(t_0) + \left[\left(\frac{\partial}{\partial x_1} \tilde{f} \right) \cdot Y \right] (c(t_0)) \\
&= \left(f \cdot \frac{\mathrm{D}}{dt} V \right)(t_0) + (f' \cdot V)(t_0),
\end{aligned}$$

which concludes the proof of lemma II.17.

Exercise 5: Lemma II.20

Let now $\varphi\colon M \to M$ be a diffeomorphism. Recall that the pushforward $\varphi_* X$ is defined as $(\varphi_* X)(p) = \left(\mathrm{D}_{\varphi^{-1}(p)}\,\varphi\right)\left(X(\varphi^{-1}(p))\right)$ for $X \in \mathrm{Vect}(M), p \in M$. The goal is to prove the following lemma.

Lemma II.20: Let ∇ be the Levi-Civita connection of a Riemannian manifold (M,g) and $\varphi \in \mathrm{Is}(X)$. Then $\nabla_{\varphi_* X}\varphi_* Y = \varphi_*(\nabla_X Y)$.

Solution:
Throughout this exercise, let $X, Y, X_1, X_2, Y_1, Y_2, Z \in \mathrm{Vect}(M), f, g \in \mathrm{C}^\infty(M), p \in M, \lambda, \mu \in \mathbb{R}$. We first want to collect some properties of $\varphi_*\colon \mathrm{Vect}(M) \to \mathrm{Vect}(M)$.

Lemma (automorphism): φ_* is a Lie-algebra-automorphism of $\mathrm{Vect}(M)$.

Proof:

- Linearity $\varphi_*(\lambda X_1 + \mu X_2) = \lambda\varphi_* X_1 + \mu\varphi_* X_2$.

$$\begin{aligned}
\varphi_*(\lambda X_1 + \mu X_2)(p) &= (\mathrm{D}_{\varphi^{-1}(p)}\,\varphi)\cdot(\lambda X_1(\varphi^{-1}(p)) + \mu X_2(\varphi^{-1}(p))) \\
&= \lambda\cdot(\mathrm{D}_{\varphi^{-1}(p)}\,\varphi)\cdot X_1(\varphi^{-1}(p)) + \mu\cdot(\mathrm{D}_{\varphi^{-1}(p)}\,\varphi)\cdot X_2(\varphi^{-1}(p)) \\
&= \lambda(\varphi_* X_1)(p) + \mu(\varphi_* X_2)(p).
\end{aligned}$$

- Inverse $(\varphi_*)^{-1} = (\varphi^{-1})_*$

$$\begin{aligned}
(\varphi_* \circ (\varphi^{-1})_* X)(p) &= (\varphi_*((\varphi^{-1})_* X))(p) \\
&= (\mathrm{D}_{\varphi^{-1}(p)}\,\varphi)\cdot((\varphi^{-1})_* X)(\varphi^{-1}(p)) \\
&= (\mathrm{D}_{\varphi^{-1}(p)}\,\varphi)\cdot(\mathrm{D}_p\,\varphi^{-1})\cdot X(p) \\
&= (\mathrm{D}_p\,\varphi\circ\varphi^{-1})\cdot X(p) = X(p)
\end{aligned}$$

- Lie brackets $\varphi_*[X,Y] = [\varphi_* X.\varphi_* Y]$.

To show that the Lie-brackets are preserved we want to use Lemma 2, which is proven a bit later. To make sense of the Lie bracket, we need to think of X, Y and $[X,Y]$ as derivations $C^\infty(M) \to C^\infty(M)$.

$$\begin{aligned}
(\varphi_*[X,Y])(f) &= \varphi_*\left([X,Y](\varphi_*^{-1} f)\right) && \text{(Lemma 2)} \\
&= \varphi_*\left(X(Y(\varphi_*^{-1} f)) - Y(X(\varphi_*^{-1} f))\right) && \text{(Def of } [\cdot,\cdot]) \\
&= \varphi_*\left(X(\varphi_*^{-1}(\varphi_* Y)(f)) - Y(\varphi_*^{-1}(\varphi_* X)f)\right) && \text{(Lemma 2)} \\
&= \varphi_*\varphi_*^{-1}\left((\varphi_* X)((\varphi_* Y)f) - (\varphi_* Y)((\varphi_* X)f)\right) && \text{(Lemma 2)} \\
&= [\varphi_* X, \varphi_* Y](f) && \text{(Def of } [\cdot,\cdot])
\end{aligned}$$

□

At various occasions we will see need to use how f interacts with φ_* and X, so we state two more lemmas. It will be convenient to use the notation of the pushforward $\varphi_* f = f \circ \varphi^{-1}$ of f.

Lemma 1: $\varphi_*(f\cdot X) = (\varphi_* f)\cdot(\varphi_* X)$.

Proof:

$$\begin{aligned}(\varphi_*(fX))(p) &= (\mathrm{D}_{\varphi^{-1}(p)}\,\varphi)\cdot f(\varphi^{-1}(p))\cdot X(\varphi^{-1}(p))\\ &= f(\varphi^{-1}(p))\cdot(\mathrm{D}_{\varphi^{-1}(p)}\,\varphi)\cdot X(\varphi^{-1}(p))\\ &= ((f\circ\varphi^{-1})\cdot\varphi_*X)(p)\\ &= ((\varphi_*f)\cdot(\varphi_*X))(p)\end{aligned}$$

□

Lemma 2: $\varphi_*(Xf) = (\varphi_*X)(\varphi_*f)$.

Proof:

$$\begin{aligned}((\varphi_*X)(\varphi_*f))(p) &= (\mathrm{D}_p\,\varphi_*f)\cdot\varphi_*X(p)\\ &= (\mathrm{D}_p\, f\circ\varphi)\cdot(\mathrm{D}_{\varphi_*^{-1}(p)}\,\varphi)\cdot X(\varphi^{-1})\\ &= (\mathrm{D}_{\varphi^{-1}(p)}\,\varphi_*f)\cdot X(\varphi^{-1}(p))\\ &= (Xf)(\varphi^{-1}(p))\\ &= (\varphi_*(Xf))(p)\end{aligned}$$

□

(1) Show that $\mathrm{D}_X\, Y = \varphi_*^{-1}\left(\nabla_{\varphi_*X}(\varphi_*Y)\right)$ is a connection.

Solution:
We have to check three conditions. First $\mathrm{C}^\infty(M)$-linearity in X

$$\begin{aligned}D_{fX}Y &= \varphi_*^{-1}\left(\nabla_{\varphi_*(fX)}(\varphi_*Y)\right) && \text{(Definition)}\\ &= \varphi_*^{-1}\left(\nabla_{(\varphi_*f)\cdot\varphi_*X}(\varphi_*Y)\right) && \text{(Lemma 1)}\\ &= \varphi_*^{-1}\left((\varphi_*f)\cdot\nabla_{\varphi_*X}(\varphi_*Y)\right) && (\mathrm{C}^\infty(M)\text{-linearity of }\nabla)\\ &= (\varphi_*^{-1}\varphi_*f)\cdot\varphi_*^{-1}\left(\nabla_{\varphi_*X}(\varphi_*Y)\right) && \text{(Lemma 1)}\\ &= fD_XY. && \text{(Definition)}\end{aligned}$$

and

$$\begin{aligned}D_{X_1+X_2}Y &= \varphi_*^{-1}\left(\nabla_{\varphi_*(X_1+X_2)}(\varphi_*Y)\right) && \text{(Definition)}\\ &= \varphi_*^{-1}\left(\nabla_{\varphi_*X_1}\varphi_*Y + \nabla_{\varphi_*X_2}\varphi_*Y\right) && (\mathrm{C}^\infty(M)\text{-linearity of }\nabla)\\ &= \varphi_*^{-1}\left(\nabla_{\varphi_*X_1}\varphi_*Y\right) + \varphi^{-1}\left(\nabla_{\varphi_*X_2}\varphi_*Y\right) && (\varphi_*\text{ is automorphism})\\ &= D_{X_1}Y + D_{X_2}Y. && \text{(Definition)}\end{aligned}$$

Second, $\mathbb{R}$-linearity in Y follows directly from $\mathbb{R}$-linearity of ∇ and φ^{-1}.
Third,

$$\begin{aligned}D_XfY &= \varphi_*^{-1}\left(\nabla_{\varphi_*X}(\varphi_*(fY))\right) && \text{(Definition)}\\ &= \varphi_*^{-1}\left(\nabla_{\varphi_*X}((\varphi_*f)\cdot\varphi_*Y)\right) && \text{(Lemma 1)}\\ &= \varphi_*^{-1}\left((\varphi_*f)\cdot\nabla_{\varphi_*X}(\varphi_*Y) + ((\varphi_*X)(\varphi_*f))\cdot\varphi_*Y\right) && (\text{Rules for }\nabla)\\ &= \varphi_*^{-1}\varphi_*f\cdot\varphi_*^{-1}\left(\nabla_{\varphi_*X}\varphi_*Y\right) + \varphi_*^{-1}\left((\varphi_*X)\,(\varphi_*f)\right)\cdot\varphi_*^{-1}(\varphi_*Y) && \text{(Lemma 1)}\\ &= f\cdot\varphi_*^{-1}\left(\nabla_{\varphi_*X}\varphi_*Y\right) + \varphi_*^{-1}\left(\varphi_*(Xf)\right)\cdot\varphi_*^{-1}(\varphi_*Y) && \text{(Lemma 2)}\\ &= f\cdot\varphi_*^{-1}\left(\nabla_{\varphi_*X}\varphi_*Y\right) + (Xf)\cdot Y\\ &= f\cdot D_XY + (Xf)\cdot Y && \text{(Definition)}\end{aligned}$$

This completes the proof that $\mathrm{D}_X Y$ is a connection.

(2) Show that $\mathrm{D}_X Y - \mathrm{D}_Y X = [X, Y]$.

Solution:

$$\begin{aligned}
\mathrm{D}_X Y - \mathrm{D}_Y X &= \varphi_*^{-1}\left(\nabla_{\varphi_* X}(\varphi_* Y)\right) - \varphi^{-1}\left(\nabla_{\varphi_* Y}(\varphi_* X)\right) \\
&= \varphi_*^{-1}\left(\nabla_{\varphi_* X}(\varphi_* Y) - \nabla_{\varphi_* Y}(\varphi_* X)\right) && \text{(linearity of } \varphi_*^{-1}\text{)} \\
&= \varphi_*^{-1}\left([\varphi_* X, \varphi_* Y]\right) && (\nabla \text{ is Levi-civita connection}) \\
&= \varphi_*^{-1}\left(\varphi_* [X, Y]\right) && (\varphi_* \text{ preserves brackets}) \\
&= [X, Y]
\end{aligned}$$

(3) Show that $X\langle Y, Z\rangle = \langle \mathrm{D}_X Y, Z\rangle + \langle Y, \mathrm{D}_X Z\rangle$.

Solution:
We first show that $\langle \varphi_* X, \varphi_* Y\rangle = \varphi_*\langle X, Y\rangle$.

$$\begin{aligned}
\langle \varphi_* X, \varphi_* Y\rangle(p) &= g_p\left(\mathrm{D}_{\varphi^{-1}(p)}\,\varphi \cdot X(\varphi^{-1}(p)), \mathrm{D}_{\varphi^{-1}(p)}\,\varphi \cdot Y(\varphi^{-1}(p))\right) \\
&= g_{\varphi^{-1}(p)}\left(X(\varphi^{-1}(p)), Y(\varphi^{-1}(p))\right) && (\varphi \text{ is an isometry}) \\
&= \langle X, Y\rangle(\varphi^{-1}(p)) = (\varphi_*\langle X, Y\rangle)(p)
\end{aligned}$$

Now

$$\begin{aligned}
&\langle \mathrm{D}_X Y, Z\rangle + \langle Y, \mathrm{D}_X Z\rangle \\
&\quad= \langle \varphi_*^{-1}\left(\nabla_{\varphi_* X}(\varphi_* Y)\right), Z\rangle + \langle Y, \varphi_*^{-1}\left(\nabla_{\varphi_* X}(\varphi_* Z)\right)\rangle \\
&\quad= \varphi_*^{-1}\langle \nabla_{\varphi_* X}(\varphi_* Y), \varphi_* Z\rangle + \varphi_*^{-1}\langle \varphi_* Y, \nabla_{\varphi_* X}(\varphi_* Z)\rangle && \text{(Remark above)} \\
&\quad= \varphi_*^{-1}(\langle \nabla_{\varphi_* X}(\varphi_* Y), \varphi_* Z\rangle + \langle \varphi_* Y, \nabla_{\varphi_* X}(\varphi_* Z)\rangle) && \text{(Linearity of } \mathrm{C}^\infty(M)\text{)} \\
&\quad= \varphi_*^{-1}(\varphi_* X\langle \varphi_* Y, \varphi_* Z\rangle) && (\nabla \text{ is a Levi-Civita-connection}) \\
&\quad= \varphi_*^{-1}((\varphi_* X)(\varphi_*\langle Y, Z\rangle)) && \text{(Remark above)} \\
&\quad= \varphi_*^{-1}\varphi_*(X\langle Y, Z\rangle) && \text{(Lemma 2)} \\
&\quad= X\langle Y, Z\rangle
\end{aligned}$$

(4) Show the lemma by using that the Levi-Civita-connection is unique.

Solution: The Levi-Civita-connection is unique, and $\mathrm{D}_X Y$ satisfies is a Levi-Civita-connection. So $\nabla_X Y = \mathrm{D}_X Y = \varphi_*^{-1}\left(\nabla_{\varphi_* X}\varphi_* Y\right)$. Applying φ_* on both sides results in

$$\varphi_*(\nabla_X Y) = \nabla_{\varphi_* X}\varphi_* Y.$$