Solution Sheet 2

Exercise 1: Prop II.22 (4)

Let $\gamma : \mathbb{R} \to M$ be a geodesic to a globally symmetric space M. Consider the transvection

$$\mathcal{T}_t^\gamma = s_{\gamma\left(\frac{t}{2}\right)} \circ s_{\gamma(0)} \in \mathrm{Is}(M)^\circ$$

along γ. For $b \in \mathbb{R}$, $\eta(t) = \gamma(t+b)$ is also a geodesic and we can define the transvection $\mathcal{T}_t^\eta$ along η. Use the fact that $t \mapsto \mathcal{T}_t$ is a 1-parameter group (Prop II.22 (3)) to show that

$$\mathcal{T}_t^\gamma = \mathcal{T}_t^\eta.$$

Solution:
We calculate:

$$\begin{aligned}
\mathcal{T}_t^\eta &= s_{\eta(t/2)} \circ s_{\eta(0)} \\
&= s_{\gamma(t/2+b)} \circ s_{\gamma(b)} \\
&= s_{\gamma(t/2+b)} \circ s_{\gamma(0)} \circ s_{\gamma(-b)} \circ s_{\gamma(0)} \\
&= s_{\gamma\left(\frac{t+2b}{2}\right)} \circ s_{\gamma(0)} \circ s_{\gamma\left(\frac{-2b}{2}\right)} \circ s_{\gamma(0)} \\
&= \mathcal{T}_{t+2b}^\gamma \circ \mathcal{T}_{-2b}^\gamma \\
&= \mathcal{T}_t^\gamma.
\end{aligned}$$

Exercise 2: A symmetric space with non-compact K.

Let

$$A = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda > 0 \right\},$$
$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\},$$

then the Iwasawa-decomposition states that the map

$$\begin{aligned}
\mathrm{SO}(2) \times A \times N &\to \mathrm{SL}(2,\mathbb{R}) \\
(k, a, n) &\mapsto kan
\end{aligned}$$

is a diffeomorphism (but not a group-homomorphism).

(1) Explain why $\pi_1(\mathrm{SL}(2,\mathbb{R})) = \mathbb{Z}$.

Solution: By the Iwasawa-decomposition we have homeomorphisms $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SO}(2) \times A \times N \cong S^1 \times \mathbb{R} \times \mathbb{R}$ and thus $\pi_1(\mathrm{SL}(2,\mathbb{R})) = \pi_1(S^1) = \mathbb{Z}$.

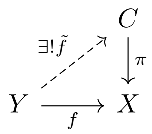
(2) Check that $\sigma\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}), g \mapsto {}^t g^{-1}$ is an involution.

Solution: Note that σ needs to be an automorphism. Being a homomorphism $\sigma(gh) = \sigma(g)\sigma(h)$ and $\sigma \circ \sigma = \mathrm{Id}$ follows directly from properties of the inverse and the transpose. Bijectivity follows from $\sigma \circ \sigma = \mathrm{Id}$. Finally, most matrices in $\mathrm{SL}(2,\mathbb{R})$ are not fixed by σ, so $\sigma \neq \mathrm{Id}$.

(3) By covering space theory we can lift σ to the universal cover $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Argue why $\tilde{\sigma}\colon \widetilde{\mathrm{SL}(2,\mathbb{R})} \to \widetilde{\mathrm{SL}(2,\mathbb{R})}$ is also an involution. You may use that the universal cover of a path-connected topological group is again a topological group.

Solution:
Recall from covering space theory the following fact:

Let $\pi\colon C \to X$ be a cover and $f\colon Y \to X$ a continuous map. Pick $y_0 \in Y$ and $c_0 \in C$, which lies over $f(y_0)$, i.e. $\pi(c_0) = f(y_0)$. If Y is simply connected, then there exists a unique lift $\tilde{f}\colon Y \to C$ with $\pi \circ \tilde{f} = f$ and $\tilde{f}(y_0) = c_0$.



In our case, $Y = C = \widetilde{\mathrm{SL}(2,\mathbb{R})}$ is the universal cover and thus simply connected. Let us write $\pi\colon \widetilde{\mathrm{SL}(2,\mathbb{R})} \to \mathrm{SL}(2,\mathbb{R})$, and $f = \sigma \circ \pi$. Fix an element $\tilde{\mathrm{Id}}$ in the universal cover with $\pi(\tilde{\mathrm{Id}}) = \mathrm{Id}$, then we get a unique map $\tilde{\sigma}\colon \widetilde{\mathrm{SL}(2,\mathbb{R})} \to \widetilde{\mathrm{SL}(2,\mathbb{R})}$ with $\tilde{\sigma}(\tilde{\mathrm{Id}}) = \tilde{\mathrm{Id}}$ (Note: $\tilde{\sigma}$ is called the lift of σ, even though strictly speaking it is the lift of $\sigma \circ \pi$).

We have to show that $\tilde{\sigma}$ is a homomorphism: For this, consider the map

$$\begin{aligned} h\colon \widetilde{\mathrm{SL}(2,\mathbb{R})} \times \widetilde{\mathrm{SL}(2,\mathbb{R})} &\to \widetilde{\mathrm{SL}(2,\mathbb{R})} \\ (g,h) \qquad &\mapsto \tilde{\sigma}(gh)^{-1}\tilde{\sigma}(g)\tilde{\sigma}(h) \end{aligned}$$

Since $\pi(gh) = \pi(g)\pi(h)$ (the multiplication in the universal covering is the lift of the multiplication in the group), π is a homomorphism. We have

$$\begin{aligned} \pi(h(g,h)) &= \pi(\tilde{\sigma}(gh)^{-1}\tilde{\sigma}(g)\tilde{\sigma}(h)) \\ &= \pi(\tilde{\sigma}(gh)^{-1})\pi(\tilde{\sigma}(h))\pi(\tilde{\sigma}(h)) \\ &= \pi(\tilde{\sigma}(gh))^{-1}\pi(\tilde{\sigma}(h))\pi(\tilde{\sigma}(h)) \\ &= \sigma(\pi(gh))^{-1}\sigma(\pi(g))\sigma(\pi(h)) \\ &= \sigma(\pi(g)\pi(h))^{-1}\sigma(\pi(g)\pi(h)) \\ &= \mathrm{Id} = \pi(\tilde{\mathrm{Id}}) \end{aligned}$$

so h is a lift of $\pi \circ h$ and so is the constant function $(g,h) \mapsto \tilde{\mathrm{Id}}$. Since the lift is unique we have $h(g,h) = \tilde{\mathrm{Id}}$, i.e. $\tilde{\sigma}(gh) = \tilde{\sigma}(g)\tilde{\sigma}(h)$.

The composition $\tilde{\sigma} \circ \tilde{\sigma}$ satisfies $\pi \circ \tilde{\sigma} \circ \tilde{\sigma} = \sigma \circ \pi \circ \tilde{\sigma} = \sigma \circ \sigma \circ \pi = \pi$, so $\tilde{\sigma} \circ \tilde{\sigma}$ as well as the constant function $g \mapsto \tilde{\mathrm{Id}}$ is a lift of π. By the uniqueness, we get that $\tilde{\sigma} \circ \tilde{\sigma}(g) = \tilde{\mathrm{Id}}$ for all $g \in \widetilde{\mathrm{SL}(2,\mathbb{R})}$. In particular, $\tilde{\sigma}$ is an automorphism.

Finally, since σ is not the identity, its lift is also not the lift of the identity, i.e. $\tilde{\sigma}$ is not the identity-map on $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. This concludes the proof that $\tilde{\sigma}$ is an involution.

(4) Prove that $\widetilde{\mathrm{SL}(2,\mathbb{R})}^{\tilde{\sigma}} = \widetilde{\mathrm{SO}(2)} \cong \mathbb{R}$.

Solution:
The map $\sigma|_{\mathrm{SO}(2)} \colon \mathrm{SO}(2) \to \mathrm{SO}(2)$ is the identity. Its lift $\tilde{\sigma}|_{\widetilde{\mathrm{SO}(2)}} \colon \widetilde{\mathrm{SO}(2)} \to \widetilde{\mathrm{SO}(2)}$ therefore also has to be the identity by uniqueness of the lift. So if $g \in \widetilde{\mathrm{SO}(2)}$, then $\tilde{\sigma}(g) = g$, i.e. $g \in \widetilde{\mathrm{SL}(2,\mathbb{R})}^{\tilde{\sigma}}$.

If on the other hand $g \in \widetilde{\mathrm{SL}(2,\mathbb{R})}$ satisfies $\tilde{\sigma}(g) = g$, then $\pi(g) = \pi(\tilde{\sigma}(g)) = \sigma(\pi(g))$, so $\pi(g) \in \mathrm{SL}(2,\mathbb{R})^{\sigma} = \mathrm{SO}(2)$. Thus $g \in \widetilde{\mathrm{SO}(2)}$.

This implies $\widetilde{\mathrm{SL}(2,\mathbb{R})}^{\tilde{\sigma}} = \widetilde{\mathrm{SO}(2)}$.

Recall that for a closed subgroup $G < \mathrm{GL}(n,\mathbb{R})$, the adjoint representation is given by

$$\begin{aligned} \mathrm{Ad}_G \colon G &\to \mathrm{GL}(\mathfrak{g}) \\ g &\mapsto (X \mapsto gXg^{-1}) \end{aligned}$$

(5) Calculate the kernel of $\mathrm{Ad}_{\mathrm{SL}(2,\mathbb{R})}|_{\mathrm{SO}(2,\mathbb{R})}$ to see that

$$\mathrm{Ad}_{\mathrm{SL}(2,\mathbb{R})}(\mathrm{SO}(2,\mathbb{R})) = {}^{\mathrm{SO}(2,\mathbb{R})}/_{\pm 1}.$$

Solution:
The elements g in the kernel satisfy $X = gXg^{-1}$ for all $X \in \mathfrak{sl}(2,\mathbb{R}) = \left\{X \in \mathbb{R}^{2\times 2} \colon \operatorname{tr}(X) = 0\right\}$. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} x & y \\ z & -x \end{pmatrix},$$

then we have $X = gXg^{-1}$ implies

$$Xg = \begin{pmatrix} ax+bz & ay-bx \\ cx+dz & cy-dx \end{pmatrix} = \begin{pmatrix} ax+cy & bx+dy \\ az-cx & bz-dx \end{pmatrix} = gX$$

so $bz = cy$ for all $z, y \in \mathbb{R}$, so $b = 0 = c$. So we have $ay = dy$ and $dz = az$ which imply $a = d$. Since $g \in \mathrm{SO}(2)$, $\det(g) = ad = a^2 = 1$. So $a = \pm 1$. We conclude that g has to be $\pm\mathrm{Id}$. And indeed both $\pm\mathrm{Id}$ are in $\mathrm{SO}(2)$. By the isomorphism-theorem we have

$$\mathrm{Ad}_{\mathrm{SL}(2,\mathbb{R})}(\mathrm{SO}(2,\mathbb{R})) \cong \mathrm{SO}(2,\mathbb{R})/\pm\mathrm{Id}.$$

(6) Argue why $\mathrm{Ad}_{\widetilde{\mathrm{SL}(2,\mathbb{R})}}(\widetilde{\mathrm{SO}(2,\mathbb{R})}) = \mathrm{Ad}_{\mathrm{SL}(2,\mathbb{R})}(\mathrm{SO}(2,\mathbb{R}))$.

Solution:
The Lie algebra $\mathfrak{g}$ only depends on a neighborhood, so

$$\mathrm{Lie}(\mathrm{SL}(2,\mathbb{R})) = \mathfrak{g} = \mathrm{Lie}\left(\widetilde{\mathrm{SL}(2,\mathbb{R})}\right).$$

Since the left-multiplication on the universal cover is the lift of the left-multiplication of $\mathrm{SL}(2,\mathbb{R})$, they can be identified in a small neighborhood around $o = \mathrm{Id}$. The adjoint representation $\mathrm{Ad}(g) = \mathrm{D}_o \mathrm{Int}(g)$ is a derivative at a point and thus also only depends on a neighborhood. We conclude that image of the adjoint representations is equal.

We set $G = \widetilde{\mathrm{SL}(2,\mathbb{R})}$ and $K = \widetilde{\mathrm{SO}(2,\mathbb{R})}$. Note that K is not compact but $\mathrm{Ad}_G(K)$ is. We therefore still get a symmetric space ${}^{G}/_{K}$.

Exercise 3: K-invariant scalar product

Let V be real vectorspace and $K < \mathrm{GL}(V)$ a compact subgroup. Prove that there exists a K-invariant scalar product on V.

Solution: Recall that for any topological group G, there exists a left-Haar measure μ, i.e. for any $f \in C^\infty(G)$, we have

$$\int_G f(gh)d\mu = \int_G f(g)d\mu.$$

Since K is compact, the Haar measure is finite $\mu(K) < \infty$. Let $\langle \cdot, \cdot \rangle$ be any scalar product on V. As $K < \mathrm{GL}(V)$, we can define

$$B(v, w) = \frac{1}{\mu(K)} \int_K \langle gv, gw \rangle d\mu$$

for $v, w \in V$. Clearly, B is bilinear and symmetric. Positive definiteness follows from the positive definiteness and continuity of $\langle \cdot, \cdot \rangle$. Using the left-invariance of the Haar-measure for $f(g) = \langle gv, gw \rangle$,

$$\begin{aligned} B(hv, hw) &= \frac{1}{\mu(K)} \int_K \langle ghv, ghw \rangle d\mu = \frac{1}{\mu(K)} \int_K f(gh) d\mu \\ &= \frac{1}{\mu(K)} \int_K f(g) d\mu = B(v, w), \end{aligned}$$

we see that B is K-invariant.

Exercise 4: The center

(1) Let G be a connected topological group and $N \lhd G$ a normal subgroup which is discrete. Show that $N \subset Z(G)$ is contained in the center $Z(G)$ of G.

Solution:
Fix $n \in N$. Consider the set $X = \left\{ gng^{-1} \colon g \in G \right\}$. It is a subset of N by construction. It is connected since it is the image of the connected set G under a continuous map. Since N is discrete, X consists of only one point. When we consider $g = e_G$, we see that X must contain n. So $X = \{n\}$. In particular $\forall g \in G \colon gng^{-1} = n$. So n lies in the center $Z(G)$.

(2) Let (G, K) be a Riemannian symmetric pair and $Z(G)$ the center of G. Show that $\mathrm{Ad}_G \colon G \to \mathrm{GL}(\mathfrak{g})$ induces an isomorphism of Lie groups:

$$K/(K \cap Z(G)) \to \mathrm{Ad}_G(K) < \mathrm{GL}(\mathfrak{g}).$$

Solution:
In the definition of Riemannian symmetric pair, G is assumed to be connected.
Let $\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation and $Z(G)$ the center.
Claim 1: $\ker \mathrm{Ad} \subset Z(G)$.
Proof:

We have the commuting diagram:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Int}(g)} & G \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{\mathrm{Ad}(g)} & \mathfrak{g} \end{array}$$

Let $g \in \ker \mathrm{Ad}$, i.e. $\mathrm{Ad}(g) = \mathrm{D}_e \,\mathrm{Int}(g) = \mathrm{Id}_{\mathfrak{g}}$. By the commuting diagram we have

$$\exp X = \exp \mathrm{Ad}(g) X = \mathrm{Int}(g) \exp X = g \exp X g^{-1}$$

for any $X \in \mathfrak{g}$. In particular, $ghg^{-1} = h$ for all $h = \exp(X)$ in a neighborhood of e. Since G is connected, by sheet 1, Exercise 3 (5), $G^\circ = G$ is generated by such a neighborhood. This means that all $h \in G$ can be written as $h = h_1 \cdot \ldots \cdot h_n$ where for all i, $gh_ig^{-1} = h_i$ and thus $ghg^{-1} = h$. □

Claim 2: $Z(G) \subset \ker \mathrm{Ad}$.
Proof:
Let $g \in Z(G)$, i.e. $ghg^{-1} = h$ for all $h \in G$. Now $\mathrm{Int}(g) = \mathrm{Id}_G$, so $\mathrm{Ad}(g) = \mathrm{D}_e \,\mathrm{Int}(g) = \mathrm{D}_e \,\mathrm{Id}_G = \mathrm{Id}_{\mathfrak{g}}$. □

The two claims imply $\ker \mathrm{Ad} = Z(G)$, so by the isomorphism theorem

$$G/Z(G) \tilde{\to} \mathrm{Ad}(G)$$

and we can restrict to K to get

$$K/(K \cap Z(G)) \tilde{\to} \mathrm{Ad}_G(K)$$