Solution to Exercise Sheet 4

Exercise 1: Compact symmetric spaces

Let $K$ be a compact connected Lie group with $\dim(K) \geq 1$. Define $G = K \times K$ and $\sigma(g, h) = (h, g)$ for $(g, h) \in G$.

(1) Show that $(G, G^\sigma)$ is a Riemannian symmetric pair with involution $\sigma$.

Solution: Since $K$ is connected, so is $G$. $G^\sigma = \{(k, k') \in G; \sigma(k, k') = (k, k')\} = \{(k, k); k \in K\}$ is a closed subgroup of $G$, which is homeomorphic to $K$. Since $K$ is compact, so is $G^\sigma$ and so is $\text{Ad}(G^\sigma)$. Since $\dim K \geq 1$, $\sigma$ is an involution and clearly $(G^\sigma)^\sigma \subset G^\sigma$. Thus $(G, G^\sigma)$ is a RSP and $G/G^\sigma$ is a Riemannian symmetric space.

(2) Consider the action $(g, h).k = gh^{-1}$ of $(g, h) \in G$ on $k \in K$. Show that $G/G^\sigma \rightarrow K$ is a homeomorphism.

Solution: We note that the action is transitive ( $(k', k) \in G$ sends $k \in K$ to $k' \in K$). Furthermore, $G$ acts by homeomorphisms on $K$ and $K$ is thus a homogeneous space. Homogeneous spaces are homeomorphic to $G/\text{Stab}_G(o)$ for any point $o$. We choose $o = e_K \in K$, then $\text{Stab}_G(e_K) = \{(g, h) \in G; (g, h).e_K = e_K\} = G^\sigma$.

Recall that by the Hopf-Rinow-theorem, the following are equivalent for a Riemannian manifold $M$:

- The closed bounded subsets of $M$ are compact.
- $M$ is complete as a metric space.
- $M$ is geodesically complete, i.e. $\forall p \in M, \text{Exp}_p : T_p M \rightarrow M$ is defined on the entire tangent space $T_p M$.

Moreover, if $M$ satisfies the above, then any two points $p, q \in M$ can be joined by a (minimal) geodesic.

(3) Use the Hopf-Rinow theorem to show that the Lie group exponential is surjective.

Solution:
Since $M$ is compact, every closed subset is compact. By Hopf-Rinow, $M$ is geodesically complete and we can find geodesics connecting any two points. This implies that the Riemannian exponential $\text{Exp}_p : T_p M \rightarrow M$ is surjective. The diagram

\[ p \xrightarrow{\exp_p} G \]
\[ T_{e_K} M \xrightarrow{\text{Exp}_{e_K}} M \]
commutes. Thus we have the surjective map
\[
\begin{align*}
\text{Lie}(K) \to & \quad \mathfrak{p} \to \quad G \to \quad K \\
X \mapsto & \quad (X, -X) \mapsto \quad (\exp(X), \exp(-X)) \mapsto \quad \exp(2X)
\end{align*}
\]
and note that precomposition with the isomorphism \( X \mapsto \frac{1}{2}X \) results in the Lie-group-exponential of \( K \), so it is surjective.

**Exercise 2: Theorem III.9: Classification of effective OSP**

Let \((g, \theta)\) be an effective orthogonal symmetric Lie-algebra. We have the Cartan decomposition \( g = u \oplus \mathfrak{c} = (\mathfrak{k} \oplus \mathfrak{p}) \). We decomposed \( \mathfrak{c} = \mathfrak{c}_0 \oplus \mathfrak{c}_+ \oplus \mathfrak{c}_- \) and defined \( u_+ = [\mathfrak{c}_+, \mathfrak{c}_+] \) and \( u_- = [\mathfrak{c}_-, \mathfrak{c}_-] \). \( u_0 \) is defined to be the orthogonal complement of \( u_+ \oplus u_- \) in \( u \).

1. Prove that \( u_- \oplus \mathfrak{c}_- \) and \( u_+ \oplus \mathfrak{c}_+ \) are ideals in \( g \).

**Solution:** Throughout the solutions, let \( \mu, \eta \in \{0, +, -\} \).

By Lemma II.12(1), \( u_\mu \) is an ideal in \( u \) and by the rest of Lemma II.12 \( [u_\mu, \mathfrak{c}_0] = 0 \), whenever \( \mu \neq \eta \). Since \( g = u \oplus \mathfrak{c} \) is the Cartan-decomposition, \( \text{ad}(u) \) preserves the decomposition. So
\[
[u_\mu, \mathfrak{c}_\mu] \subset \mathfrak{c}_\mu.
\]

We need one more fact from the proof of Lemma III.10, namely
\[
[\mathfrak{c}_0, \mathfrak{c}] = 0
\]
We calculate
\[
[g, u_\mu + \mathfrak{c}_\mu] = [\mathfrak{c}, u_\mu] + [\mathfrak{c}, \mathfrak{c}_\mu] + [u, u_\mu] + [u, \mathfrak{c}_\mu]
\]
\[
\subset \mathfrak{c}_\mu + ([\mathfrak{c}_0, \mathfrak{c}_\mu] + [\mathfrak{c}_+, \mathfrak{c}_\mu] + [\mathfrak{c}_-, \mathfrak{c}_\mu]) + [u, u_\mu] + \mathfrak{c}_\mu
\]
\[
\subset \mathfrak{c}_\mu + ([\mathfrak{c}_0, \mathfrak{c}_\mu] + [\mathfrak{c}_+, \mathfrak{c}_\mu] + [\mathfrak{c}_-, \mathfrak{c}_\mu]) + u_\mu + \mathfrak{c}_\mu
\]
\[
= u_\mu + \mathfrak{c}_\mu + [\mathfrak{c}_0, \mathfrak{c}_\mu] + [\mathfrak{c}_-, \mathfrak{c}_\mu] + u_\mu + \mathfrak{c}_\mu
\]
\[
\subset u_\mu + \mathfrak{c}_\mu + u_\mu
\]
(1)

2. Prove that \( u_0 \oplus \mathfrak{c}_0, u_- \oplus \mathfrak{c}_- \) and \( u_+ \oplus \mathfrak{c}_+ \) are \( \theta \)-stable and pairwise orthogonal with respect to \( B_\theta \).

**Solution:** For elements \( X \in u \), we have \( \theta(X) = X \) and for \( Y \in \mathfrak{c} \), we have \( \theta(Y) = -Y \). Thus any subvector space of \( u \) and \( \mathfrak{c} \) is \( \theta \)-stable and also direct products of them.

For \( \mu \neq \eta \), we have
\[
B_\theta(\mathfrak{c}_\mu, u_\mu, \mathfrak{c}_\eta, u_\eta) = B_\theta(\mathfrak{c}_\mu, \mathfrak{c}_\eta) + B_\theta(\mathfrak{c}_\mu, u_\eta) + B_\theta(\mathfrak{c}_\eta, u_\mu) + B_\theta(u_\mu, u_\eta),
\]
where the middle two are 0, since \( u \) and \( \mathfrak{c} \) are orthogonal by Lemma III.6(1). \( B_\theta(\mathfrak{c}_0, \mathfrak{c}_0) = 0 \) since \( \mathfrak{c} = \mathfrak{c}_0 \oplus \mathfrak{c}_+ \oplus \mathfrak{c}_- \) is orthogonal (we chose a ONB in the definition). For \( \mu, \eta \neq 0 \), we have Lemma III.11, that shows \( B_\theta(u_\mu, u_\eta) = 0 \). If \( \mu = 0 \) or \( \eta = 0 \), then the Killing form is also 0, since \( u_0 \) was defined to be the orthogonal complement of \( u_+ \oplus u_- \) in \( u \).
(3) Find an OSL $(\mathfrak{g}, \theta)$, such that $c_0 = 0$, but $u_0 \neq 0$.

Solution: The idea is to have a large $u$ and a small $c$. This means that $\theta$ should fix lots of points. For example one can take $\theta = \theta_{\mathfrak{sl}(2,\mathbb{R})} \times \mathfrak{so}(3)$, where $\theta_{\mathfrak{sl}(2,\mathbb{R})} = D_\epsilon \sigma$ (for $\sigma(g) = g^{-1}$) is the usual Cartan-involution on $\mathfrak{sl}(2,\mathbb{R})$. Then $u = E_1 \theta = \mathfrak{t} \times \mathfrak{so}(3)$ and $c = E_{-1} \theta = p \times 0$, where $\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{t} \oplus p$ is the Cartan-decomposition of $\mathfrak{sl}(2,\mathbb{R})$.

We need to check that $(\mathfrak{g}, \theta)$ is an orthogonal symmetric Lie-algebra (OSL): $\theta$ is an involutive automorphism since $\theta_{\mathfrak{sl}(2,\mathbb{R})}$ and $\mathfrak{Id}_{\mathfrak{so}(3)}$ are and we also have $\theta \neq \mathfrak{Id}_\mathfrak{g}$. The definition of OSL requires $u$ to be compactly-embedded in $\mathfrak{g}$, i.e. $\mathfrak{ad}_\mu(u)$ is the Lie-Algebra of a compact subgroup of $GL(\mathfrak{g})$. This is true since $\mathfrak{t} \times \mathfrak{so}(3)$ is the lie algebra of the compact group $SO(2) \times SO(3) < SL(2,\mathbb{R}) \times SO(3) < GL(\mathfrak{g})$. Note that we were forced to take the Lie-algebra of a compact group as the second factor $(\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ would not have worked, but $\mathfrak{so}(3) \times \mathfrak{so}(3)$ would have).

Now one can calculate the Killing form

$$A = \begin{pmatrix}
-8 & 0 & 0 \\
0 & 8 & 0 \\
0 & 0 & 8 \\
-2 & 0 & 0 \\
0 & 0 & -2 \\
0 & 0 & -2
\end{pmatrix}$$

in the basis

$$e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, e_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of $\mathfrak{g} = \mathfrak{t} \times \mathfrak{p} \times \mathfrak{p} \times \mathfrak{so}(3) = \langle e_1 \rangle \oplus \langle e_2, e_3 \rangle \oplus \langle e_4, e_5, e_6 \rangle$. For the first factor $\mathfrak{sl}(2,\mathbb{R})$ this was done in the third exercise class. The second factor $\mathfrak{so}(3)$ is of compact type and therefore all eigenvalues are negative. Following the definitions we get the decomposition of $c = c_0 \oplus c_+ \oplus c_- = 0 \oplus \mathfrak{p} \oplus 0$.

Now $u_+ := [e_+, e_+] = \mathfrak{t} \times 0$ and $u_- := [e_-, e_-] = 0$. The remaining orthogonal complement is $u_0 = 0 \times \mathfrak{so}(3) \neq 0$. So we have found an OSL with $c_0 = 0$ and $u_0 \neq 0$ and thus it is necessary to make the distinction when defining the $\mathfrak{g}_\mu$ for $\mu \in \{0, +, -\}$.

(4) Let $\mathfrak{n} \subset \mathfrak{g}$ be an ideal of a Lie-algebra $\mathfrak{g}$. Prove that $B_n = B_{|\mathfrak{n}| \times \mathfrak{n}}$.

Solution: Let us write a basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ of $\mathfrak{g}$, where $e_1, \ldots, e_n$ is a basis of $\mathfrak{n}$. Since $\mathfrak{n}$ is an ideal, for $X \in \mathfrak{n}, Z \in \mathfrak{g}$, we have $[X, Z] \in \mathfrak{n}$. Therefore $\mathfrak{ad}_g(X)$ is of the form

$$\mathfrak{ad}_g(X) = \begin{pmatrix} \mathfrak{ad}_\mathfrak{n}(X) & * \\ 0 & 0 \end{pmatrix}$$
and so for \( X, Y \in \mathfrak{n} \) we have
\[
B_\mathfrak{g}(X,Y) = \text{tr}(ad_\mathfrak{g}(X) \circ ad_\mathfrak{g}(Y)) \\
= \text{tr} \begin{pmatrix} 
\text{ad}_\mathfrak{n}(X) \circ \text{ad}_\mathfrak{n}(Y) \\
0 \\
0 
\end{pmatrix} \\
= \text{tr}(\text{ad}_\mathfrak{n}(X) \circ \text{ad}_\mathfrak{n}(Y)) \\
= B_\mathfrak{n}(X,Y).
\]

(5) Find an example of a subalgebra \( \mathfrak{n} \subseteq \mathfrak{g} \), such that \( B_\mathfrak{n} \neq B_\mathfrak{g}|_{\mathfrak{n} \times \mathfrak{n}} \).

\textit{Solution:} We consider the Cartan-decomposition \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p} \) and set \( \mathfrak{n} := \mathfrak{k} \). We know that \( \mathfrak{k} \) is a subalgebra of \( \mathfrak{g} \). Taking the basis
\[
e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
we get
\[
ad_\mathfrak{g}(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}
\]
(for a derivation see the third exercise class). Let \( X = \lambda_1 \cdot e_1, Y = \lambda_2 \cdot e_1 \in \mathfrak{n} = \langle e_1 \rangle \). Then
\[
B_\mathfrak{g}(X,Y) = \text{tr}(ad_\mathfrak{g}(X) \circ ad_\mathfrak{g}(X)) = \lambda_1 \cdot \lambda_2 \cdot \text{tr} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix} = -8 \cdot \lambda_1 \cdot \lambda_2,
\]
but
\[
B_\mathfrak{n}(X,Y) = \text{tr}(ad_\mathfrak{n}(X) \circ ad_\mathfrak{n}(X)) = \text{tr}(0 \cdot 0) = 0
\]
since \( \text{ad}_\mathfrak{n}(X) = 0 \) because \([e_1, e_1] = 0\).

(6) Let \( \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \) a direct sum of two ideals \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \). Further let \( \mathfrak{k}_1 \) and \( \mathfrak{k}_2 \) be subalgebras of \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \). Show that \( \mathfrak{k}_1 + \mathfrak{k}_2 \) is compactly embedded in \( \mathfrak{g} \) if and only if \( \mathfrak{k}_1 \) and \( \mathfrak{k}_2 \) is compactly embedded in \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \).

This implies that \( u_0, u_- \), \( u_+ \) are compactly embedded in \( g_0, g_- \) and \( g_+ \).

\textit{Hint: For connected} \( G \) \textit{and} \( K \prec G \), \textit{there is an isomorphism}
\[
K/(K \cap Z(G)) \cong \text{Ad}_G(K)
\]
\textit{(compare Sheet 2, exercise 4(2)). Use} \( \text{Lie}(\text{Ad}_G(K)) = \text{ad}_{\text{Lie}(G)}(\text{Lie}(K)) \).

\textit{Solution:} By Lie’s third theorem, there exist connected, (and simply connected) Lie groups \( G_1 \) and \( G_2 \) with \( \text{Lie}(G_1) = \mathfrak{g}_1 \) and \( \text{Lie}(G_2) = \mathfrak{g}_2 \). The Lie group \( G := G_1 \times G_2 \) satisfies \( \text{Lie}(G) = \mathfrak{g}_1 \times \mathfrak{g}_2 \). Since \( \mathfrak{k}_1 \) and \( \mathfrak{k}_2 \) are Lie-subalgebras, there exist \( K_1 \) and \( K_2 \) Lie-subgroups of \( G_1 \) and \( G_2 \) with \( \text{Lie}(K_1) = \mathfrak{k}_1 \) and \( \text{Lie}(K_2) = \mathfrak{k}_2 \). We also have \( K := K_1 + K_2 \) with \( \text{Lie}(K) = \mathfrak{k}_1 \times \mathfrak{k}_2 \).

Now we have the center \( Z(G) = Z(G_1) \times Z(G_2) \) and
\[
Z(G) \cap K = (Z(G_1) \times Z(G_2)) \cap (K_1 \times K_2) = (Z(G_1) \cap K_1) \times (Z(G_2) \cap K_2),
\]
so
\[ \text{Ad}_G(K) = K / (Z(G) \cap K) = (K_1 \times K_2) / (Z(G_1) \cap K_1 \times Z(G_2) \cap K_2) = K_1 / (Z(G_1) \cap K_1) \times K_2 / (Z(G_2) \cap K_2) = \text{Ad}_{G_1}(K_1) \times \text{Ad}_{G_2}(K_2). \]

Now \( \text{ad}_g(\mathfrak{t}_1 + \mathfrak{t}_2), \text{ad}_g(\mathfrak{t}_1) \) and \( \text{ad}_g(\mathfrak{t}_2) \) are the Lie-algebras of the groups \( \text{Ad}_G(K) \), \( \text{Ad}_{G_1}(K_1) \) and \( \text{Ad}_{G_2}(K_2) \). So \( \mathfrak{t}_1 + \mathfrak{t}_2 \) is compactly embedded in \( \mathfrak{g} \) by definition if and only if \( \text{Ad}(K) \) is compact which is equivalent to saying \( \text{Ad}_{G_1}(K_1) \) and \( \text{Ad}_{G_2}(K_2) \) are compact, i.e. both \( \mathfrak{t}_1 \) and \( \mathfrak{t}_2 \) are compactly embedded in \( \mathfrak{g}_1 \) resp. \( \mathfrak{g}_2 \).

**Exercise 3:** **Theorem III.19:** Classification of s.c. RSS

(1) Let \( H, N \leq G \) be two normal subgroups. Show that \([N,H] \subset N \cap H\).

**Solution:** Let \( n(hn^{-1}h^{-1}) \in [N,H] \), then \((nhn^{-1})h^{-1} \in HH^{-1} \subset H \) and \( n(hn^{-1}h^{-1}) \in nN \subset N. \) So \( nhn^{-1}h^{-1} \in H \cap N. \)

(2) Let \( H, N \lhd G \) be connected subgroups. Show that \([N,H] \) is a connected subgroup of \( G \).

**Solution:** The map \([\cdot,\cdot] : N \times H \to G \) is continuous, since it is a composition of multiplications. The image of connected sets under a continuous map is connected.

(3) Show that \((\tilde{G}_0, K_0), (\tilde{G}_+, K_+) \) and \((\tilde{G}_-, K_-) \) are Riemannian symmetric pairs.

**Solution:** Let \( \mu \in \{0,+,\} \). The \( \tilde{G}_\mu \) can be assumed to be connected. In the proof of theorem III.19, we have that \( \psi|_{K_0 \times K_- \times K_+} : K_0 \times K_- \times K_+ \to p^{-1}(K) \) is a homeomorphism. The product of sets is closed if and only if all the factors are closed, so \( K_\mu \) are closed subgroups of \( \tilde{G} \) and therefore also of \( \tilde{G}_\mu \). Since \( \mathfrak{t}_\mu \) are compactly embedded, we get that \( \text{Ad}_{\tilde{G}_\mu}(K_\mu) \) are compact.

By the Lie-group-correspondence, since \( \tilde{G} \) is simply connected we get \( \sigma : \tilde{G} \to \tilde{G} \) a unique Lie-group automorphism, such that \( D_\sigma \sigma = \theta. \) Now (using the pullback of the isomorphism \( \psi \)), we can restrict \( \sigma \) to \( \sigma_\mu : \tilde{G}_\mu \to \tilde{G}_\mu \). Since \( \theta_\mu : g_\mu \to g_\mu^{-1} \) is an involution, so is \( \sigma_\mu \) (they are not the identity).

It remains to show that \( (\tilde{G}_\mu^o)^o \subset K_\mu \subset \tilde{G}_\mu^o \). Let \( X \in \mathfrak{t}_\mu \). Then \( \exp(X) \in \tilde{G}_\mu \). We have that \( \sigma_\mu(\exp(X)) = \exp(\theta_\mu X) = \exp(X) \). So for all \( g \in K_\mu \) in a small neighborhood of \( e \), we have \( \sigma_\mu(g) = g \). Since a neighborhood generates the connected group \( K_\mu \), we can write elements \( g \in K_\mu \) as a product \( g = g_1 \cdot \ldots \cdot g_n \) and we get \( \sigma_\mu(g) = \sigma_\mu(g_1) \cdot \ldots \cdot \sigma_\mu(g_n) = g \). So \( K_\mu \subset \tilde{G}_\mu^o \).

Now we consider a neighborhood \( V \subset \exp(\mathfrak{g}_\mu) \) of \( e \) of \( \tilde{G}_\mu \). Let \( \exp(tX) \in V \cap (\tilde{G}_\mu^o)^o \) for \( t \in (-\varepsilon, \varepsilon) \). Then \( \exp(tX) = \sigma_\mu(\exp(tX)) = \exp(t\theta_\mu(X)), \) so...
(taking the derivative) we get $X = \theta_\mu(X)$, i.e. $X \in \mathfrak{t}_\mu$ and thus $V \cap (\tilde{G}_\mu^\sigma)^{\circ} \subset K_\mu$. Now since $(\tilde{G}_\mu^\sigma)^{\circ}$ is connected, the elements are generated by elements in $K_\mu$, i.e. $(\tilde{G}_\mu^\sigma)^{\circ} \subset K_\mu$.

We conclude that $(\tilde{G}_\mu, K_\mu)$ are Riemannian symmetric pairs for $\{0, -, +\}$. 