Solution to Exercise Sheet 4

Exercise 1: Compact symmetric spaces

Let K be a compact connected Lie group with $\dim(K) \geq 1$. Define $G = K \times K$ and $\sigma(g,h) = (h,g)$ for $(g,h) \in G$.

(1) Show that (G, G^σ) is a Riemannian symmetric pair with involution σ.

Solution: Since K is connected, so is G. $G^\sigma = \{(k,k') \in G \colon \sigma(k,k') = (k,k')\} = \{(k,k) \colon k \in K\}$ is a closed subgroup of G, which is homeomorphic to K. Since K is compact, so is G^σ and so is $\mathrm{Ad}(G^\sigma)$. Since $\dim K \geq 1$, σ is an involution and clearly $(G^\sigma)^\circ \subset G^\circ$. Thus (G, G^σ) is a RSP and G/G^σ is a Riemannian symmetric space.

(2) Consider the action $(g,h).k = gkh^{-1}$ of $(g,h) \in G$ on $k \in K$. Show that

$$G/G^\sigma \to K$$

is a homeomorphism.

Solution: We note that the action is transitive ($(k',k) \in G$ sends $k \in K$ to $k' \in K$). Furthermore, G acts by homeomorphisms on K and K is thus a homogeneous space. Homogeneous spaces are homeomorphic to $G/\operatorname{Stab}_G(o)$ for any point o. We choose $o = e_K \in K$, then $\operatorname{Stab}_G(e_K) = \{(g,h) \in G \colon (g,h).e_K = e_K\} = G^\sigma$.

Recall that by the Hopf-Rinow-theorem, the following are equivalent for a Riemannian manifold M:

- The closed bounded subsets of M are compact.
- M is complete as a metric space.
- M is geodesically complete, i.e. $\forall p \in M, \operatorname{Exp}_p \colon T_pM \to M$ is defined on the entire tangent space T_pM.

Moreover, if M satisfies the above, then any two points $p, q \in M$ can be joined by a (minimal) geodesic.

(3) Use the Hopf-Rinow theorem to show that the Lie group exponential is surjective.

Solution:
Since M is compact, every closed subset is compact. By Hopf-Rinow, M is geodesically complete and we can find geodesics connecting any two points. This implies that the Riemannian exponential $\operatorname{Exp}_p \colon T_pM \to M$ is surjective. The diagram

$$\begin{tikzcd} \mathfrak{p} \arrow[r, "\exp|_{\mathfrak{p}}"] \arrow[d, "\sim"] & G \arrow[d, "\pi"] \\ T_{e_K}M \arrow[r, "\operatorname{Exp}_{e_K}"'] & M \end{tikzcd}$$

commutes. Thus we have the surjective map

$$\begin{array}{ccccccc} \mathrm{Lie}(K) \to & \mathfrak{p} & \to & G & \to & K \\ X \mapsto & (X,-X) & \mapsto & (\exp(X),\exp(-X)) \mapsto & & \exp(2X) \end{array}$$

and note that precomposition with the isomorphism $X \mapsto \frac{1}{2}X$ results in the Lie-group-exponential of K, so it is surjective.

Exercise 2: Theorem III.9: Classification of effective OSP

Let $(\mathfrak{g},\theta)$ be an effective orthogonal symmetric Lie-algebra. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ $(= \mathfrak{k} \oplus \mathfrak{p})$. We decomposed $\mathfrak{e} = \mathfrak{e}_0 \oplus \mathfrak{e}_+ \oplus \mathfrak{e}_-$ and defined $\mathfrak{u}_+ = [\mathfrak{e}_+,\mathfrak{e}_+]$ and $\mathfrak{u}_- = [\mathfrak{e}_-,\mathfrak{e}_-]$. $\mathfrak{u}_0$ is defined to be the orthogonal complement of $\mathfrak{u}_+ \oplus \mathfrak{u}_-$ in $\mathfrak{u}$.

(1) Prove that $\mathfrak{u}_- \oplus \mathfrak{e}_-$ and $\mathfrak{u}_+ \oplus \mathfrak{e}_+$ are ideals in $\mathfrak{g}$.

Solution: Throughout the solutions, let $\mu,\eta \in \{0,+,-\}$.

By Lemma II.12(1), $\mathfrak{u}_\mu$ is an ideal in $\mathfrak{u}$ and by the rest of Lemma II.12 $[\mathfrak{u}_\mu,\mathfrak{e}_\eta] = 0$, whenever $\mu \neq \eta$. Since $\mathfrak{g} = \mathfrak{u}\oplus\mathfrak{e}$ is the Cartan-decomposition, $\mathrm{ad}(\mathfrak{u})$ preserves the decomposition. So

$$[\mathfrak{u}_\mu,\mathfrak{e}_\mu] \subset \mathfrak{e}_\mu. \tag{1}$$

We need one more fact from the proof of Lemma III.10, namely

$$[\mathfrak{e}_0,\mathfrak{e}] = 0 \tag{2}$$

We calculate

$$\begin{aligned} [\mathfrak{g},\mathfrak{u}_\mu + \mathfrak{e}_\mu] &= [\mathfrak{e},\mathfrak{u}_\mu] + [\mathfrak{e},\mathfrak{e}_\mu] + [\mathfrak{u},\mathfrak{u}_\mu] + [\mathfrak{u},\mathfrak{e}_\mu] \\ &\subset \mathfrak{e}_\mu + ([\mathfrak{e}_0,\mathfrak{e}_\mu] + [\mathfrak{e}_+,\mathfrak{e}_\mu] + [\mathfrak{e}_-,\mathfrak{e}_\mu]) + [\mathfrak{u},\mathfrak{u}_\mu] + \mathfrak{e}_\mu && (1) \\ &\subset \mathfrak{e}_\mu + ([\mathfrak{e}_0,\mathfrak{e}_\mu] + [\mathfrak{e}_+,\mathfrak{e}_\mu] + [\mathfrak{e}_-,\mathfrak{e}_\mu]) + \mathfrak{u}_\mu + \mathfrak{e}_\mu && (\mathfrak{u}_\mu \text{ is an ideal in } \mathfrak{u}) \\ &= \mathfrak{u}_\mu + \mathfrak{e}_\mu + [\mathfrak{e}_0,\mathfrak{e}_\mu] + [\mathfrak{e}_\mu,\mathfrak{e}_\mu] && (\text{Lemma III.10(3)}) \\ &\subset \mathfrak{u}_\mu + \mathfrak{e}_\mu + 0 + \mathfrak{u}_\mu && (2 \text{ and def of } \mathfrak{u}_\mu) \end{aligned}$$

(2) Prove that $\mathfrak{u}_0\oplus\mathfrak{e}_0$, $\mathfrak{u}_-\oplus\mathfrak{e}_-$ and $\mathfrak{u}_+\oplus\mathfrak{e}_+$ are θ-stable and pairwise orthogonal with respect to $B_\mathfrak{g}$.

Solution: For elements $X \in \mathfrak{u}$, we have $\theta(X) = X$ and for $Y \in \mathfrak{e}$, we have $\theta(Y) = -Y$. Thus any subvectorspace of $\mathfrak{u}$ and $\mathfrak{e}$ are θ-stable and also direct products of them.

For $\mu \neq \eta$, we have

$$B_\mathfrak{g}(\mathfrak{e}_\mu + \mathfrak{u}_\mu, \mathfrak{e}_\eta + \mathfrak{u}_\eta) = B_\mathfrak{g}(\mathfrak{e}_\mu,\mathfrak{e}_\eta) + B_\mathfrak{g}(\mathfrak{e}_\mu,\mathfrak{u}_\eta) + B_\mathfrak{g}(\mathfrak{u}_\mu,\mathfrak{e}_\eta) + B_\mathfrak{g}(\mathfrak{u}_\mu,\mathfrak{u}_\eta),$$

where the middle two are 0, since $\mathfrak{u}$ and $\mathfrak{e}$ are orthogonal by Lemma III.6(1). $B_\mathfrak{g}(\mathfrak{e}_\mu,\mathfrak{e}_\eta) = 0$ since $\mathfrak{e} = \mathfrak{e}_0 \oplus \mathfrak{e}_+ \oplus \mathfrak{e}_-$ is orthogonal (we chose a ONB in the definition). For $\mu,\eta \neq 0$, we have Lemma III.11, that shows $B_\mathfrak{g}(\mathfrak{u}_\mu,\mathfrak{u}_\eta) = 0$. If $\mu = 0$ or $\eta = 0$, then the Killing form is also 0, since $\mathfrak{u}_0$ was defined to be the orthogonal completment of $\mathfrak{u}_+ \oplus \mathfrak{u}_-$ in $\mathfrak{u}$.

(3) Find an OSL $(\mathfrak{g}, \theta)$, such that $\mathfrak{e}_0 = 0$, but $\mathfrak{u}_0 \neq 0$.

Solution: The idea is to have a large $\mathfrak{u}$ and a small $\mathfrak{e}$. This means that θ should fix lots of points. For example one can take $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{so}(3)$ and define $\theta = \theta_{\mathfrak{sl}(2,\mathbb{R})} \times \mathrm{Id}_{\mathfrak{so}(3)}$, where $\theta_{\mathfrak{sl}(2,\mathbb{R})} = \mathrm{D}_e\,\sigma$ (for $\sigma(g) = {}^t g^{-1}$) is the usual Cartan-involuion on $\mathfrak{sl}(2,\mathbb{R})$. Then $\mathfrak{u} = E_1\theta = \mathfrak{k} \times \mathfrak{so}(3)$ and $\mathfrak{e} = E_{-1}\theta = \mathfrak{p} \times 0$, where $\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan-decomposition of $\mathfrak{sl}(2,\mathbb{R})$.

We need to check that $(\mathfrak{g}, \theta)$ is an orthogonal symmetric Lie-algebra (OSL): θ is an involutive automorphism since $\theta_{\mathfrak{sl}(2,\mathbb{R})}$ and $\mathrm{Id}_{\mathfrak{so}(3)}$ are and we also have $\theta \neq \mathrm{Id}_{\mathfrak{g}}$. The definition of OSL requires $\mathfrak{u}$ to be compactly-embedded in $\mathfrak{g}$, i.e. $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{u})$ is the Lie-Algebra of a compact subgroup of $\mathrm{GL}(\mathfrak{g})$. This is true since $\mathfrak{k} \times \mathfrak{so}(3)$ is the lie algebra of the compact group $\mathrm{SO}(2) \times \mathrm{SO}(3) < \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(3) < \mathrm{GL}(\mathfrak{g})$. Note that we were forced to take the Lie-algebra of a compact group as the second factor ($\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ would not have worked, but $\mathfrak{so}(3) \times \mathfrak{so}(3)$ would have).

Now one can calculate the Killing form

$$A = \begin{pmatrix} -8 & 0 & 0 & & & \\ 0 & 8 & 0 & & 0 & \\ 0 & 0 & 8 & & & \\ & & & -2 & 0 & 0 \\ & 0 & & 0 & -2 & 0 \\ & & & 0 & 0 & -2 \end{pmatrix}$$

in the basis

$$e_1 = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0 \right), e_2 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right), e_3 = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0 \right)$$
$$e_4 = \left(0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right), e_5 = \left(0, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right), e_6 = \left(0, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

of $\mathfrak{g} = \mathfrak{k} \times 0 \oplus \mathfrak{p} \times 0 \oplus 0 \times \mathfrak{so}(3) = \langle e_1 \rangle \oplus \langle e_2, e_3 \rangle \oplus \langle e_4, e_5, e_6 \rangle$. For the first factor $\mathfrak{sl}(2,\mathbb{R})$ this was done in the third exercise class. The second factor $\mathfrak{so}(3)$ is of compact type and therefore all eigenvalues are negative. Following the definitions we get the decomposition of $\mathfrak{e} = \mathfrak{e}_0 \oplus \mathfrak{e}_+ \oplus \mathfrak{e}_- = 0 \oplus \mathfrak{p} \times 0 \oplus 0$.

Now $\mathfrak{u}_+ := [\mathfrak{e}_+, \mathfrak{e}_+] = \mathfrak{k} \times 0$ and $\mathfrak{u}_- := [\mathfrak{e}_-, \mathfrak{e}_-] = 0$. The remaining orthogonal complement is $\mathfrak{u}_0 = 0 \times \mathfrak{so}(3) \neq 0$. So we have found a OSL with $\mathfrak{e}_0 = 0$ and $\mathfrak{u}_0 \neq 0$ and thus it is necessary to make the distinction when defining the $\mathfrak{g}_\mu$ for $\mu \in \{0, +, -\}$.

(4) Let $\mathfrak{n} \lhd \mathfrak{g}$ be an ideal of a Lie-algebra $\mathfrak{g}$. Prove that $B_{\mathfrak{n}} = B_{\mathfrak{g}}|_{\mathfrak{n}\times\mathfrak{n}}$.

Solution: Let us write a basis $e_1, \dots, e_n, e_{n+1}, \dots, e_m$ of $\mathfrak{g}$, where $e_1, \dots e_n$ is a basis of $\mathfrak{n}$. Since $\mathfrak{n}$ is an ideal, for $X \in \mathfrak{n}, Z \in \mathfrak{g}$, we have $[X, Z] \in \mathfrak{n}$. Therefore $\mathrm{ad}_{\mathfrak{g}}(X)$ is of the form

$$\mathrm{ad}_{\mathfrak{g}}(X) = \begin{pmatrix} \mathrm{ad}_{\mathfrak{n}}(X) & * \\ 0 & 0 \end{pmatrix}$$

and so for $X, Y \in \mathfrak{n}$ we have

$$\begin{aligned} B_{\mathfrak{g}}(X,Y) &= \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(X)\circ\operatorname{ad}_{\mathfrak{g}}(Y)) \\ &= \operatorname{tr}\begin{pmatrix}\operatorname{ad}_{\mathfrak{n}}(X)\circ\operatorname{ad}_{\mathfrak{n}}(Y) & * \\ 0 & 0\end{pmatrix} \\ &= \operatorname{tr}(\operatorname{ad}_{\mathfrak{n}}(X)\circ\operatorname{ad}_{\mathfrak{n}}(Y)) \\ &= B_{\mathfrak{n}}(X,Y). \end{aligned}$$

(5) Find an example of a subalgebra $\mathfrak{n} \subset \mathfrak{g}$, such that $B_{\mathfrak{n}} \neq B_{\mathfrak{g}}|_{\mathfrak{n}\times\mathfrak{n}}$.

Solution: We consider the Cartan-decomposition $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) = \mathfrak{k}\oplus\mathfrak{p}$ and set $\mathfrak{n} := \mathfrak{k}$. We know that $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$. Taking the basis

$$e_1 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, e_2 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, e_3 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

we get

$$\operatorname{ad}_{\mathfrak{g}}(e_1) = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -2\\ 0 & 2 & 0\end{pmatrix}$$

(for a derivation see the third exercise class). Let $X = \lambda_1 \cdot e_1, Y = \lambda_2 \cdot e_1 \in \mathfrak{n} = \langle e_1\rangle$. Then

$$B_{\mathfrak{g}}(X,Y) = \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(X)\circ\operatorname{ad}_{\mathfrak{g}}(X)) = \lambda_1\cdot\lambda_2\cdot\operatorname{tr}\begin{pmatrix}0 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -4\end{pmatrix} = -8\cdot\lambda_1\cdot\lambda_2,$$

but

$$B_{\mathfrak{n}}(X,Y) = \operatorname{tr}(\operatorname{ad}_{\mathfrak{n}}(X)\circ\operatorname{ad}_{\mathfrak{n}}(X)) = \operatorname{tr}(0\cdot 0) = 0$$

since $\operatorname{ad}_{\mathfrak{n}}(X) = 0$ because $[e_1, e_1] = 0$.

(6) Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ a direct sum of two ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Further let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be subalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Show that $\mathfrak{k}_1 + \mathfrak{k}_2$ is compactly embedded in $\mathfrak{g}$ if and only if $\mathfrak{k}_1$ and $\mathfrak{k}_2$ is compactly embedded in $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

This implies that $\mathfrak{u}_0, \mathfrak{u}_-, \mathfrak{u}_+$ are compactly embedded in $\mathfrak{g}_0, \mathfrak{g}_-$ and $\mathfrak{g}_+$.

Hint: For connected G and $K < G$, there is an isomorphism

$$K/(K\cap Z(G)) \cong \operatorname{Ad}_G(K)$$

(compare Sheet 2, exercise 4(2)). Use $\operatorname{Lie}(\operatorname{Ad}_G(K)) = \operatorname{ad}_{\operatorname{Lie}(G)}(\operatorname{Lie}(K))$.

Solution: By Lie's third theorem, there exist connected, (and simply connected) Lie groups G_1 and G_2 with $\operatorname{Lie}(G_1) = \mathfrak{g}_1$ and $\operatorname{Lie}(G_2) = \mathfrak{g}_2$. The Lie group $G := G_1 \times G_2$ satisfies $\operatorname{Lie}(G) = \mathfrak{g}_1 \times \mathfrak{g}_2$. Since $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are Lie-subalgebras, there exist K_1 and K_2 Lie-subgroups of G_1 and G_2 with $\operatorname{Lie}(K_1) = \mathfrak{k}_1$ and $\operatorname{Lie}(K_2) = \mathfrak{k}_2$. We also have $K := K_1 + K_2$ with $\operatorname{Lie}(K) = \mathfrak{k}_1 \times \mathfrak{k}_2$.

Now we have the center $Z(G) = Z(G_1) \times Z(G_2)$ and

$$Z(G)\cap K = (Z(G_1)\times Z(G_2))\cap(K_1\times K_2) = (Z(G_1)\cap K_1)\times(Z(G_2)\cap K_2),$$

so

$$\begin{aligned}\mathrm{Ad}_G(K) &= K/(Z(G)\cap K)\\ &= (K_1\times K_2)/(Z(G_1)\cap K_1\times Z(G_2)\cap K_2)\\ &= K_1/(Z(G_1)\cap K_1)\times K_2/(Z(G_2)\cap K_2)\\ &= \mathrm{Ad}_{G_1}(K_1)\times \mathrm{Ad}_{G_2}(K_2).\end{aligned}$$

Now $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k}_1+\mathfrak{k}_2), \mathrm{ad}_{\mathfrak{g}_1}(\mathfrak{k}_1)$ and $\mathrm{ad}_{\mathfrak{g}_2}(\mathfrak{k}_2)$ are the Lie-algebras of the groups $\mathrm{Ad}_G(K)$, $\mathrm{Ad}_{G_1}(K_1)$ and $\mathrm{Ad}_{G_2}(K_2)$.

So $\mathfrak{k}_1+\mathfrak{k}_2$ is compactly embedded in $\mathfrak{g}$ by definition if and only if $\mathrm{Ad}(K)$ is compact which is equivalent to saying $\mathrm{Ad}_{G_1}(K_1)$ and $\mathrm{Ad}_{G_2}(K_2)$ are compact, i.e. both $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are compactly embedded in $\mathfrak{g}_1$ resp. $\mathfrak{g}_2$.

Exercise 3: Theorem III.19: Classification of s.c. RSS

(1) Let $H, N \triangleleft G$ be two normal subgroups. Show that $[N,H]\subset N\cap H$.

Solution: Let $nhn^{-1}h^{-1}\in[N,H]$, then $(nhn^{1})h^{-1}\in Hh^{-1}\subset H$ and $n(hn^{-1}h^{-1})\in nN\subset N$. So $nhn^{-1}h^{-1}\in H\cap N$.

(2) Let $H,N<G$ be connected subgroups. Show that $[N,H]$ is a connected subgroup of G.

Solution: The map $[\cdot,\cdot]\colon N\times H\to G$ is continuous, since it is a composition of multiplications. The image of connected sets under a continuous map is connected.

Let M be a simply connected Riemannian symmetric space. Then $\mathfrak{g} = \mathrm{Lie}(\mathrm{Is}(M)^\circ) = \mathfrak{g}_0\oplus\mathfrak{g}_+\oplus\mathfrak{g}_-$. We get corresponding Lie-subgroups G_0, G_+, G_- and their universal covers $\tilde G_0, \tilde G_+, \tilde G_-$. Let K_0, K_+, K_- be the Lie-subgroups associated to $\mathfrak{k}_0, \mathfrak{k}_+, \mathfrak{k}_-$, which come from the Cartan-decomposition of $\mathfrak{g}_0, \mathfrak{g}_+, \mathfrak{g}_-$.

(3) Show that $(\tilde G_0, K_0), (\tilde G_+, K_+)$ and $(\tilde G_-, K_-)$ are Riemannian symmetric pairs.

Solution: Let $\mu\in\{0,+,-\}$. The $\tilde G_\mu$ can be assumed to be connected. In the proof of theorem III.19, we have that $\tilde\psi|_{K_0\times K_-\times K_+}\colon K_0\times K_-\times K_+\to p^{-1}(K)$ is a homeomorphism. The product of sets is closed if and only if all the factors are closed, so K_μ are closed subgroups of $\tilde G$ and therefore also of $\tilde G_\mu$. Since $\mathfrak{k}_\mu$ are compactly embedded, we get that $\mathrm{Ad}_{\tilde G_\mu}(K_\mu)$ are compact.

By the Lie-group-correspondence, since $\tilde G$ is simply connected we get $\sigma\colon\tilde G\to\tilde G$ a unique Lie-group automorphism, such that $\mathrm{D}_e\,\sigma=\theta$. Now (using the pullback of the isomorphism ψ), we can restrict σ to $\sigma_\mu\colon\tilde G_\mu\to\tilde G_\mu$. Since $\theta_\mu\colon\ \mathfrak{g}_\mu\to\mathfrak{g}_\mu$ is an involution, so is σ_μ (they are not the identity).

It remains to show that $(\tilde G_\mu^{\sigma_\mu})^\circ\subset K_\mu\subset\tilde G_\mu^{\sigma_\mu}$. Let $X\in\mathfrak{k}_\mu$. Then $\exp(X)\in\tilde G_\mu$. We have that $\sigma_\mu(\exp(X))=\exp(\theta_\mu X)=\exp(X)$. So for all $g\in K_\mu$ in a small neighborhood of e, we have $\sigma_\mu(g)=g$. Since a neighborhood generates the connected group K_μ, we can write elements $g\in K_\mu$ as a product $g=g_1\cdot\ldots\cdot g_n$ and we get $\sigma_\mu(g)=\sigma_\mu(g_1)\cdot\ldots\cdot\sigma_\mu(g_n)=g$. So $K_\mu\subset\tilde G_\mu^{\sigma_\mu}$.

Now we consider a neighborhood $V\subset\exp(\mathfrak{g}_\mu)$ of e of $\tilde G\mu$. Let $\exp(tX)\in V\cap(\tilde G_\mu^{\sigma_\mu})^\circ$ for $t\in(-\varepsilon,\varepsilon)$. Then $\exp(tX)=\sigma_\mu(\exp(tX))=\exp(t\theta_\mu(X))$, so (taking the derivative) we get $X = \theta_\mu(X)$, i.e. $X \in \mathfrak{k}_\mu$ and thus $V \cap (\tilde{G}_\mu^{\sigma_\mu})^\circ \subset K_\mu$. Now since $(\tilde{G}_\mu^{\sigma_\mu})^\circ$ is connected, the elements are generated by elements in K_μ, i.e. $(\tilde{G}_\mu^{\sigma_\mu})^\circ \subset K_\mu$.

We conclude that $(\tilde{G}_\mu, K_\mu)$ are Riemannian symmetric pairs for $\{0, -, +\}$.