Solution to Exercise Sheet 5

Exercise 1: Theorem III.22: Decomposition into irreducible parts

Let $(\mathfrak{g}, \theta)$ be a reduced OSL with Cartan decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$. We can write

$$\mathfrak{e} = \mathfrak{e}_0 \oplus \bigoplus_{j=1}^{a} \mathfrak{p}_j,$$

where each $\mathfrak{p}_j$ is U-invariant and irreducible. We define $\mathfrak{g}_j = [\mathfrak{p}_j, \mathfrak{p}_j] + \mathfrak{p}_j$.

(1) Show that $\mathfrak{g}_j$ is a θ-stable ideal in $\mathfrak{g}$ for $1 \leq j \leq a$.

Solution: The Cartan-decomposition implies that $\theta(X) = -X$ for all $X \in \mathfrak{p}_j \subset \mathfrak{e}$. Similarly $[\mathfrak{p}_j, \mathfrak{p}_j] \subset \mathfrak{u}$ and therefore $\theta([\mathfrak{p}_j, \mathfrak{p}_j]) = [\mathfrak{p}_j, \mathfrak{p}_j]$ and thus $\theta(\mathfrak{g}_j) = \mathfrak{g}_j$.
Claim 1: $[\mathfrak{u}, \mathfrak{g}_j] \subset \mathfrak{g}_j$.
Proof: $\mathfrak{p}_j$ is U-invariant, which means that $[\mathfrak{u}, \mathfrak{p}_j] \subset \mathfrak{p}_j$. Together with the Jacobi-identity this results in the required inclusion:

$$\begin{aligned}
[\mathfrak{u}, \mathfrak{g}_j] &= [\mathfrak{u}, [\mathfrak{p}_j, \mathfrak{p}_j] + \mathfrak{p}_j] \\
&= [\mathfrak{u}, [\mathfrak{p}_j, \mathfrak{p}_j]] + [\mathfrak{u}, \mathfrak{p}_j] \\
&= -[\mathfrak{p}_j, [\mathfrak{p}_j, \mathfrak{u}]] - [\mathfrak{p}_j, [\mathfrak{u}, \mathfrak{p}_j]] + \mathfrak{p}_j \\
&\subset [\mathfrak{p}_j, \mathfrak{p}_j] + \mathfrak{p}_j = \mathfrak{g}_j \,.
\end{aligned}$$

□

Claim 2: $[\mathfrak{p}_j, \mathfrak{p}_i \oplus \mathfrak{e}_0] = 0$ for $i \neq j$.
Proof: Let $X \in \mathfrak{p}_j$ and $Y \in \mathfrak{p}_i \oplus \mathfrak{e}_0$. We note that $[X, Y] \in [\mathfrak{e}, \mathfrak{e}] \subset \mathfrak{u}$. We can take a look at the function $\mathfrak{u} \to \mathbb{R}$, defined by

$$\begin{aligned}
Z \mapsto B_{\mathfrak{g}}([X, Y], Z) &= -B_{\mathfrak{g}}(Y, [X, Z]) \\
&= -\langle AY, [X, Z] \rangle \\
&\in -\langle \mathfrak{p}_i \oplus \mathfrak{e}_0, \mathfrak{p}_j \rangle = 0,
\end{aligned}$$

where we used $\operatorname{ad} \mathfrak{g}$-invariance of $B_{\mathfrak{g}}$, Claim 1 and the orthogonality of the decomposition $\mathfrak{e} = \mathfrak{e}_0 \oplus \bigoplus_{k=1}^{a} \mathfrak{p}_k$. Now $B_{\mathfrak{g}}|\mathfrak{u} \oplus \mathfrak{u} << 0$ is negative definite (Lemma III.6) and $Z \mapsto B_{\mathfrak{g}}([X, Y], Z)$ is the zero map, which means that $[X, Y] = 0$. □
Now we see that

$$\begin{aligned}
[\mathfrak{g}, \mathfrak{g}_j] &= [\mathfrak{u}, \mathfrak{g}_j] + [\mathfrak{e}_0 \oplus \bigoplus_{i=1}^{a} \mathfrak{p}_i, \mathfrak{g}_j] \\
&\subset \mathfrak{g}_j + [\mathfrak{e}_0 \oplus \bigoplus_{i \neq j} \mathfrak{p}_i, [\mathfrak{p}_j, \mathfrak{p}_j] + \mathfrak{p}_j] + [\mathfrak{p}_j, [\mathfrak{p}_j, \mathfrak{p}_j] + \mathfrak{p}_j] && \text{(Claim 1)} \\
&= \mathfrak{g}_j + [\mathfrak{e}_0 \oplus \bigoplus_{i \neq j} \mathfrak{p}_i, [\mathfrak{p}_j, \mathfrak{p}_j]] + [\mathfrak{p}_j, [\mathfrak{p}_j, \mathfrak{p}_j]] + [\mathfrak{p}_j, \mathfrak{p}_j] && \text{(Claim 2)} \\
&\subset \mathfrak{g}_j - [\mathfrak{p}_j, [\mathfrak{p}_j, \mathfrak{e}_0 \oplus \bigoplus_{i \neq j} \mathfrak{p}_i]] - [\mathfrak{p}_j, [\mathfrak{e}_0 \oplus \bigoplus_{i \neq j} \mathfrak{p}_i, \mathfrak{p}_j]] + [\mathfrak{p}_j, \mathfrak{u}] && \text{(Jacobi, } [\mathfrak{e}, \mathfrak{e}] \subset \mathfrak{u}) \\
&\subset \mathfrak{g}_j - [\mathfrak{p}_j, \mathfrak{u}] - [\mathfrak{p}_j, \mathfrak{u}] + \mathfrak{g}_j \subset \mathfrak{g}_j && \text{(Claim 1)}
\end{aligned}$$

which concludes the proof that $\mathfrak{g}_j$ is a θ-stable ideal in $\mathfrak{g}$.

(2) From (1) and Sheet 4, exercise 2(4) it follows that $B_{\mathfrak{g}_j} = B_{\mathfrak{g}}|_{\mathfrak{g}_j \times \mathfrak{g}_j}$. Show that $B_{\mathfrak{g}_j}$ is non-degenerate, i.e. $\mathfrak{g}_j$ is semi-simple.

Solution: We have to show that if $X \in \mathfrak{g}_j$ satisfies $B_{\mathfrak{g}_j}(X, \mathfrak{g}_j) = 0$, then $X = 0$. Since $B_{\mathfrak{g}_j}$ is bilinear, we can check it in two cases:
Case 1: Let $X \in \mathfrak{p}_j \subset \mathfrak{g}_j$ with $B_{\mathfrak{g}_j}(X, \mathfrak{g}_j) = 0$. In particular $0 = B_{\mathfrak{g}_j}(X, X) = \langle AX, X\rangle = \langle c_j X, X\rangle = c_j \cdot \langle X, X\rangle$ with $c_j \neq 0$, so $X = 0$.
Case 2: Let $X \in [\mathfrak{p}_j, \mathfrak{p}_j]$ with $B_{\mathfrak{g}_j}(X, \mathfrak{g}_j) = 0$. We decompose $\mathfrak{u} = [\mathfrak{p}_j, \mathfrak{p}_j] \oplus \mathfrak{u}'$ orthogonally with respect to $B_{\mathfrak{g}}|_{\mathfrak{u} \times \mathfrak{u}} << 0$. Now we have

$$B_{\mathfrak{g}}(X, \mathfrak{u}) = B_{\mathfrak{g}}(X, [\mathfrak{p}_j, \mathfrak{p}_j]) + B_{\mathfrak{g}}(X, \mathfrak{u}') = 0 + 0,$$

and since $B_{\mathfrak{g}}|_{\mathfrak{u} \times \mathfrak{u}} << 0$, we get that $X = 0$.

(3) Let

$$\mathfrak{m} = \bigoplus_{j=1}^{a} \mathfrak{g}_j \,.$$

Prove that the centralizer

$$\mathfrak{g}_0 := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{m}) = \{X \in \mathfrak{g} \colon [X, m] = 0 \ \forall m \in \mathfrak{m}\}$$

is a θ-stable ideal in $\mathfrak{g}$.

Solution. By (1), $\mathfrak{g}_j$ (and direct products thereof) are θ-stable, so let $m \in \mathfrak{m}$ and $X \in \mathfrak{g}_0$. Then $[\theta(X), m] = \theta([X, \theta(m)]) = \theta([X, m]) = \theta(0) = 0$. So $\mathfrak{g}_0$ is θ-stable.
To see that $\mathfrak{g}_0$ is an ideal in $\mathfrak{g}$ let $m \in \mathfrak{m}$,

$$\begin{aligned}[[\mathfrak{g}_0, \mathfrak{g}], m] &= -[[\mathfrak{g}, m], \mathfrak{g}_0] - [[m, \mathfrak{g}_0], \mathfrak{g}] \\ &\subset -[\mathfrak{m}, \mathfrak{g}_0] - [0, \mathfrak{g}] = 0\end{aligned}$$

(4) Show that this gives rise to an orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{m}$.

Solution: We need to show that the orthogonal complement of $\mathfrak{m}$ is $\mathfrak{g}_0$. We claim that the orthogonal complement

$$\mathfrak{m}^{\perp} := \{X \in \mathfrak{g} \colon B_{\mathfrak{g}}(X, \mathfrak{m}) = 0\}$$

is an ideal in $\mathfrak{g}$. Indeed:

$$B_{\mathfrak{g}}([\mathfrak{g}, \mathfrak{m}^{\perp}], \mathfrak{m}) = -B_{\mathfrak{g}}(\mathfrak{m}^{\perp}, [\mathfrak{g}, \mathfrak{m}]) \subset -B_{\mathfrak{g}}(\mathfrak{m}^{\perp}, \mathfrak{m}) = 0.$$

Both $\mathfrak{m}$ and $\mathfrak{m}^{\perp}$ are ideals in $\mathfrak{g}$, so $[\mathfrak{m}^{\perp}, \mathfrak{m}] \subset \mathfrak{m} \cap \mathfrak{m}^{\perp} = 0$, so $\mathfrak{m}^{\perp} \subset \mathfrak{g}_0$.
On the other hand, the map $\mathrm{ad}(\mathfrak{g}_0) \circ \mathrm{ad}(\mathfrak{m})$ sends X to

$$\begin{aligned}[\mathfrak{g}_0, [\mathfrak{m}, X]] &= -[\mathfrak{m}, [X, \mathfrak{g}_0]] - [X, [\mathfrak{g}_0, \mathfrak{m}]] \\ &\subset -[\mathfrak{m}, \mathfrak{g}_0] - [X, \mathfrak{g}_0 \cap \mathfrak{m}] \\ &\subset -\mathfrak{g}_0 \cap \mathfrak{m} - [X, 0] = 0\end{aligned}$$

since both $\mathfrak{g}_0$ and $\mathfrak{m}$ are ideals (by (3) and (1)) in $\mathfrak{g}$ and $\mathfrak{g}_0 \cap \mathfrak{m} = 0$ by definition of $\mathfrak{g}_0$. Therefore $B_{\mathfrak{g}}(\mathfrak{g}_0, \mathfrak{m}) = \mathrm{tr}(\mathrm{ad}(\mathfrak{g}_0) \circ \mathrm{ad}(\mathfrak{m})) = 0$, so $\mathfrak{g}_0 \subset \mathfrak{m}^{\perp}$. This concludes the proof that $\mathfrak{g}_0 = \mathfrak{m}^{\perp}$.

(5) Show that $(\mathfrak{g}_0, \theta|_{\mathfrak{g}_0})$ is a reduced OSL of Euclidean type (or 0).

Consider $\mathfrak{u}_0 = \{X \in \mathfrak{g}_0 \colon \theta(X) = X\}$. By (4), the effective OSL is a orthogonal product $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{m}$ with $\mathfrak{u} \subset \mathfrak{g}$ compactly embedded. By Sheet 4, exercise 2(6), also $\mathfrak{u}_0$ is compactly embedded. If $\mathfrak{g}_0 \neq 0$, then θ is an involution and $\mathfrak{g}_0$ is an OSL.
Assume that $\mathfrak{i} \subset \mathfrak{u}_0$ is an ideal in $\mathfrak{g}_0$. Then $[\mathfrak{i}, \mathfrak{g}] = [\mathfrak{i}, \mathfrak{g}_0] + \sum_j [\mathfrak{i}, \mathfrak{g}_j] \subset \mathfrak{i} + 0$, so $\mathfrak{i} \subset \mathfrak{u}_0 \subset \mathfrak{u}$ is also an ideal in $\mathfrak{g}$. Since $(\mathfrak{g}, \theta)$ is reduced, $\mathfrak{u}$ contains no non-trivial ideals in $\mathfrak{g}$, so $\mathfrak{i}$ is trivial. This means that $\mathfrak{g}_0$ is reduced.
Let $X \in \mathfrak{u}$. Then $B_{\mathfrak{g}}([\mathfrak{e}_0, \mathfrak{e}_0], X) = -B_{\mathfrak{g}}(\mathfrak{e}_0, [\mathfrak{e}_0, X]) = \langle A\,\mathfrak{e}_0, [\mathfrak{e}_0, X]\rangle = 0$ (by the definition of $\mathfrak{e}_0$). Since $B_{\mathfrak{g}}|_{\mathfrak{u}\times\mathfrak{u}} << 0$, we get $[\mathfrak{e}_0, \mathfrak{e}_0] = 0$. Now $[\mathfrak{e}_0, \mathfrak{g}_0] = [\mathfrak{e}_0, \mathfrak{e}_0] + [\mathfrak{e}_0, \mathfrak{u}_0] \subset 0 + \mathfrak{u}_e$ implies that $\mathfrak{e}_0$ is an abelian ideal in $\mathfrak{g}_0$, i.e $(\mathfrak{g}_0, \theta|_{\mathfrak{g}_0 \times \mathfrak{g}_0})$ is of Euclidean type.

(6) Show that the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{j=1}^{a} \mathfrak{g}_j$ is unique up to reordering.

Hint: Consider two decompositions and the projection $\pi\colon \mathfrak{g} \to \mathfrak{g}_0$. Show that the projection of a semi-simple ideal is a semi-simple ideal. Use that the Killing form of an ideal is the restriction of $B_{\mathfrak{g}}$ to the ideal.

Solution: Consider two decompositions

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{i=1}^{a} \mathfrak{g}_i = \mathfrak{g}_0' \oplus \bigoplus_{k=1}^{a'} \mathfrak{g}_k'$$

(orthogonal wrt $B_{\mathfrak{g}}$), where the $\mathfrak{g}_i$ and $\mathfrak{g}_k'$ are irreducible ideals in $\mathfrak{g}$. Define $\pi\colon \mathfrak{g} \to \mathfrak{g}_0$ to be the orthogonal projection to $\mathfrak{g}_0$.

Lemma 1: If $\mathfrak{n} \subset \mathfrak{g}$ is a semi-simple ideal in $\mathfrak{g}$, then $\pi(\mathfrak{n})$ is also a semi-simple ideal.

Proof: For $X \in \mathfrak{n}$, we can write uniquely $X = \pi(X) + Y$ for $Y \in \mathfrak{m} := \bigoplus_{i=1}^{a} \mathfrak{g}_i$. Let $Z \in \mathfrak{g}$, then $[X, Z] = [\pi(X) + Y, Z] = [\pi(X), Z] + [Y, Z]$ is the decomposition into $\mathfrak{g}_0$ and $\mathfrak{m}$, because $[\pi(X), Z] \in \mathfrak{g}_0$ and $[Y, Z] \in \mathfrak{m}$. This implies $[\pi(X), Z] = \pi([X, Z]) \in \pi(\mathfrak{n})$, since $\mathfrak{n}$ is an ideal in $\mathfrak{g}$. This shows that $\pi(\mathfrak{n})$ is an ideal.
Let $\pi(X) \in \pi(\mathfrak{n})$. Assume that for every $Y \in \mathfrak{g}$, $B_{\mathfrak{g}}(\pi(x), \pi(Y)) = 0$. Note that we used $B_{\mathfrak{n}} = B_{\mathfrak{g}}|_{\mathfrak{n}\times\mathfrak{n}}$ since $\mathfrak{n}$ is an ideal in $\mathfrak{g}$ (see sheet 4, exercise 2(4)). We can write $Y = \pi(Y) + Y'$, where $\pi(Y)$ and Y' are orthogonal wrt $B_{\mathfrak{g}}$, in fact Y' is orthogonal to all of $\pi(\mathfrak{n})$, in particular $B(\pi(X), Y') = 0$. So we have that for all $Y \in \mathfrak{n}$,

$$B_{\mathfrak{g}}(\pi(X), Y) = B_{\mathfrak{g}}(\pi(X), \pi(Y)) + B_{\mathfrak{g}}(\pi(X), Y') = 0.$$

Since $\mathfrak{n}$ is semi-simple, $\pi(X) = 0$. This concludes the proof that $\pi(\mathfrak{n})$ is semi-simple. $\square$

Now we apply Lemma 1 to the semi-simple ideal $\mathfrak{m}' = \bigoplus_{k=1}^{a'} \mathfrak{g}_k'$ of $\mathfrak{g}$ to get that $\pi(\mathfrak{m}') \subset \mathfrak{g}_0$ is a semi-simple ideal in $\mathfrak{g}$. Since $\mathfrak{g}_0$ is of Euclidean type, we have that $\mathfrak{p}_0 := \mathfrak{g}_0 \cap \mathfrak{p}$ is an abelian ideal in $\mathfrak{g}_0$. So semi-simplicity of $\pi(\mathfrak{m}')$ implies $\pi(\mathfrak{m}') \cap \mathfrak{p}_0 = 0$. Thus also $[\pi(\mathfrak{m}'), \mathfrak{p}_0] = 0$, so $\pi(\mathfrak{m}') \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{p}_0) = \mathfrak{p}_0$ and thus $\pi(\mathfrak{m}') = 0$. We have shown that $\mathfrak{m}' \subset \mathfrak{g}_0^{\perp} = \mathfrak{m}$. Doing the same with the projection $\mathfrak{g} \to \mathfrak{g}_0'$, results in $\mathfrak{m} \subset \mathfrak{m}'$, so $\mathfrak{m} = \mathfrak{m}'$ and $\mathfrak{g}_0 = \mathfrak{g}_0'$.

Now we have to show that the semi-simple part of the decomposition is unique (up to permutation). We define $\mathfrak{p}_i = \mathfrak{g}_i \cap \mathfrak{p}$ and $\mathfrak{p}'_k = \mathfrak{g}'_k \cap \mathfrak{p}$ for $i \in \{1, \dots, a\}$ and $k \in \{1, \dots, a'\}$. We define $\pi_i \colon \mathfrak{g} \to \mathfrak{p}_i$. We note that $\mathrm{ad}(\mathfrak{k})$ and the π_i commute:

Lemma 2: For all $X \in \mathfrak{k}, Y \in \mathfrak{p}$, we have $\pi_i([X, Y]) = [X, \pi_i(Y)]$.

Proof: We can write $Y = \sum_{i=0}^{a} Y_i$ uniquely, where $Y_i = \pi_i(Y) \in \mathfrak{p}_i$. Then we can also write

$$[X, Y] = \left[X, \sum_{i=0}^{a} Y_i\right] = \sum_{j=0}^{a} [X, Y_i]$$

uniquely, where $[X, Y_j] \in \mathfrak{g}_j \cap \mathfrak{p} = \mathfrak{p}_j$ because $\mathfrak{g}_j$ is an ideal and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Therefore $\pi_i([X, Y]) = [X, Y_i] = [X, \pi_i(Y)]$. $\square$

Now look at a $\mathfrak{p}'_k$ and its image $\pi_i(\mathfrak{p}'_k) \subset \mathfrak{p}_i$. Note that $\mathfrak{g}'_k$ and thus $\mathfrak{p}'_k$ are $\mathrm{ad}(\mathfrak{k})$-invariant. By Lemma 2, $\mathrm{ad}(\mathfrak{k})(\pi_i(\mathfrak{p}'_k)) = \pi_i(\mathrm{ad}(\mathfrak{k})(\mathfrak{p}'_k)) \subset \pi_i(\mathfrak{p}'_k)$, i.e. $\pi_i(\mathfrak{p}'_k)$ is $\mathrm{ad}(\mathfrak{k})$-invariant. Since $\mathfrak{p}_i$ is irreducible, the only invariant subspaces are 0 and $\mathfrak{p}_i$, so we $\pi_i(\mathfrak{p}'_k) = 0$ or $\pi_i(\mathfrak{p}'_k) = \mathfrak{p}_i$. We know that $\mathfrak{p} = \bigoplus_i \mathfrak{p}_i = \bigoplus_k \mathfrak{p}'_k$, so for every $\mathfrak{p}'_k$, there exists an i such that $\pi_i(\mathfrak{p}'_k) = \mathfrak{p}_i$. To get a bijection between the factors, we have to make sure that for any k there is at most one such i.

Lemma 3: If $\pi_i(\mathfrak{p}'_k) = \mathfrak{p}_i$, then there is a vectorspace-isomorphism $\mathfrak{p}'_k \to \mathfrak{p}_j$ given by π_i.

Proof: The map $\pi_i|_{\mathfrak{p}'_k} \colon \mathfrak{p}'_k \to \mathfrak{p}_i$ is surjective by assumption. Let $Y \in \ker(\pi_i|_{\mathfrak{p}'_k})$, i.e. $Y \in \mathfrak{p}'_k$ with $\pi_i(Y) = 0$. By lemma 2, $\pi_i([\mathfrak{k}, Y]) = [\mathfrak{k}, \pi_i(Y)] = 0$, which means that the kernel is $\mathrm{ad}(\mathfrak{k})$-invariant. Since $\mathfrak{p}'_k$ is irreducible, the kernel has to be 0 or all of $\mathfrak{p}'_k$. Since $\mathfrak{p}_i \neq 0$ and $\pi_i|_{\mathfrak{p}'_k}$ is surjective, the kernel cannot be all of $\mathfrak{p}'_k$, so it has to be 0. We conclude that $\pi_i|_{\mathfrak{p}'_k}$ is an isomorphism. $\square$

Assume there are i, j with $\pi_i(\mathfrak{p}'_k) = \mathfrak{p}_i$ and $\pi_j(\mathfrak{p}'_k) = \mathfrak{p}_j$. Then using Lemma 3, we get a vectorspace-isomorphism $\varphi \colon \mathfrak{p}_i \to \mathfrak{p}_j$, which to $Y_i \in \mathfrak{p}_i$ finds a $Y_k \in \mathfrak{p}'_k$ with $\pi_i(Y_k) = Y_i$ and $\pi_j(Y_k) = \varphi(Y_i)$, in particular $\pi_j = \varphi \circ \pi_i$. Considering Lemma 2, we get

$$\begin{aligned} \mathrm{ad}(\mathfrak{k}) \circ \varphi(Y_i) &= [\mathfrak{k}, \pi_j(Y_k)] = \pi_j([\mathfrak{k}, Y_k]) = \varphi \circ \pi_i([\mathfrak{k}, Y_k]) \\ &= \varphi([\mathfrak{k}, \pi_i(Y_k)]) = \varphi([\mathfrak{k}, Y_i]) = \varphi \circ \mathrm{ad}(\mathfrak{k})(Y_i). \end{aligned}$$

To show that $i = j$, we have to study the $\mathrm{ad}(\mathfrak{k})$-action on $\mathfrak{p}$ in more detail:

Lemma 4: For a fixed i, we have that the kernel of the $\mathrm{ad}(\mathfrak{k}_i)$-action on $\mathfrak{p}$ is

$$\ker(\mathrm{ad}(\mathfrak{k}_i)) = \{Y \in \mathfrak{p} \colon [\mathfrak{k}_i, Y] = 0\} = \bigoplus_{j \neq i} \mathfrak{p}_j \,.$$

Proof: We first claim that $\mathrm{ad}(\mathfrak{k}_i)$ acts non-trivially on $\mathfrak{p}_i$, in fact, for all $X \in \mathfrak{k}_i$, $\mathrm{ad}(X) \colon \mathfrak{p}_i \to \mathfrak{p}_i$ is $\equiv 0$ if and only if $X = 0$.
The direction $X = 0 \Rightarrow \mathrm{ad}(X) \equiv 0$ is immediate. For " $\Leftarrow$ ", let $\mathrm{ad}(X)|_{\mathfrak{p}_i} \equiv 0$. For all $Y \in \mathfrak{p}_i$ we get the relations $[X, Y] = 0, [X, \mathfrak{k}_i] \subset \mathfrak{k}_i, [Y, \mathfrak{p}_i] \subset \mathfrak{k}_i, [Y, \mathfrak{k}_i] \subset \mathfrak{p}_i$, which allow us to write as matrices

$$\mathrm{ad}(X) = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \quad \text{and} \quad \mathrm{ad}(Y) = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix},$$

where we decomposed $\mathfrak{g}_i = \mathfrak{p}_i \oplus \mathfrak{k}_i$. Now

$$B_{\mathfrak{g}}(X,Y) = \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Y)) = \operatorname{tr}\left(\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}\right) = \operatorname{tr}\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} = 0$$

for all $Y \in \mathfrak{p}_i$ and since $\mathfrak{g}_i$ is an ideal, $B_{\mathfrak{g}_i} = B_{\mathfrak{g}}|_{\mathfrak{g}_i \times \mathfrak{g}_i}$, so we can use semi-simplicity of $\mathfrak{g}_i$ to see that $X = 0$.
Now we know that $\operatorname{ad}(\mathfrak{k}_i)$ is non-zero on $\mathfrak{p}_i$. Note that for $i \neq j$, we have $\operatorname{ad}(\mathfrak{k}_i)(\mathfrak{p}_j) = [\mathfrak{k}_i, \mathfrak{p}_j] \subset [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_i \cap \mathfrak{g}_j = 0$. This shows the stated form of the kernel of the $\operatorname{ad}(\mathfrak{k}_i)$-action on $\mathfrak{p}$. $\square$

Assume there are $i \neq j$, such that $\pi_i(\mathfrak{p}'_k) = \mathfrak{p}_i$ and $\pi_j(\mathfrak{p}'_k) = \mathfrak{p}_j$. By Lemma 4, $\operatorname{ad}(\mathfrak{k}_i)|_{\mathfrak{p}_j} \equiv 0$. So let $Y_i \in \mathfrak{p}_i \setminus \{0\}$ with $\operatorname{ad}(\mathfrak{k}_i)(Y_i) \neq 0$. Then

$$0 \neq \varphi \circ \operatorname{ad}(\mathfrak{k}_i)(Y_i) = \operatorname{ad}(\mathfrak{k}_i) \circ \varphi(Y_i) = 0$$

results in a contradiction. (Essentially we have used that even though two different $\mathfrak{p}_i, \mathfrak{p}_j$ may be isomorphic as vectorspaces, when you view them as $\operatorname{ad}(\mathfrak{k})$-modules, they are no longer isomorphic.)
This means that to any $\mathfrak{p}'_k$, exactly one space $\mathfrak{p}_i = \mathfrak{p}_j$ is associated. We conclude that $a = a'$ and $\{\mathfrak{p}_i\}_{i=1}^a = \{\mathfrak{p}_k\}_{k=1}^a$.
Finally, since $\mathfrak{g}_i = \mathfrak{p}_i \oplus \mathfrak{k}_i$ and $\mathfrak{g}'_k = \mathfrak{p}'_k \oplus \mathfrak{k}'_k$, we also get that $\{\mathfrak{g}_i\}_{i=1}^a = \{\mathfrak{g}'_k\}_{k=1}^a$.
We have shown that the factors are unique up to reordering.

Exercise 2: Lemma IV.10: Automorphism group

Let $\mathfrak{g}$ be a real Lie-algebra. Prove that $\operatorname{Aut}(\mathfrak{g})$ is a Lie-subgroup of $\operatorname{GL}(\mathfrak{g})$ with Lie-algebra $\operatorname{Der}(\mathfrak{g})$.

Hint: The Lie algebra of a Lie-subgroup $H < G$ is

$$\operatorname{Lie}(H) = \{X \in \operatorname{Lie}(G) \colon \forall t \in \mathbb{R} \colon \exp(tX) \in H\}.$$

Solution: If we choose a basis $\{e_1, \ldots, e_n\}$ for $\mathfrak{g}$, we can identify $\operatorname{GL}(\mathfrak{g}) = \operatorname{GL}(n)$. Then by linearity

$$\begin{aligned}\operatorname{Aut}(\mathfrak{g}) &= \{g \in \operatorname{GL}(\mathfrak{g}) \colon \forall X, Y \in \mathfrak{g} \colon g([X,Y]) = [g(X), g(Y)]\} \\ &= \{g \in \operatorname{GL}(n) \colon \forall i, j \colon g([e_i, e_j]) = [g(e_i), g(e_j)]\},\end{aligned}$$

which is a closed subgroup of $\operatorname{GL}(n)$. Closed subgroups of Lie-groups are Lie-groups, so $\operatorname{Aut}(\mathfrak{g})$ is a Lie-subgroup of $\operatorname{GL}(\mathfrak{g})$.

Let $\delta \in \operatorname{Lie}(\operatorname{Aut}(\mathfrak{g}))$. Then $\exp(t\delta) \in \operatorname{Aut}(\mathfrak{g})$, so for all $X, Y \in \mathfrak{g}$,

$$\exp(t\delta)([X,Y]) = [\exp(t\delta)(X), \exp(t\delta)(Y)]. \tag{1}$$

Since we are working with matrix groups, we can write $\exp(t\delta) = \operatorname{Id} + t\delta + \frac{1}{2}t^2\delta^2 + \ldots$ and taking the derivative results in

$$\begin{aligned}\delta([X,Y]) &= \frac{d}{dt}\bigg|_{t=0} [X,Y] + t\delta([X,Y]) + \frac{1}{2}t^2\delta^2([X,Y]) + \ldots \\ &= \frac{d}{dt}\bigg|_{t=0} [X + t\delta(X) + \frac{1}{2}t^2\delta^2(X) + \ldots, Y + t\delta(Y) + \frac{1}{2}t^2\delta^2(Y) + \ldots] \\ &= \frac{d}{dt}\bigg|_{t=0} [X,Y] + t \cdot ([\delta(X), Y] + [X, \delta(Y)]) + t^2 \cdot (\ldots) + \ldots \\ &= [\delta(X), Y] + [X, \delta(Y)].\end{aligned}$$

So if $\delta \in \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$ then $\delta \in \mathrm{Der}(\mathfrak{g})$.
Now let $\delta \in \mathrm{Der}(\mathfrak{g})$. We need to show that $\exp(\delta) \in \mathrm{Aut}(\mathfrak{g})$. To use the definition of exp for matrix-groups, we first need to understand $\delta^n([X,Y])$. We prove by induction

$$\delta^n([X,Y]) = \sum_{i=0}^{n} \binom{n}{i} [\delta^i(X), \delta^{n-i}Y].$$

The formula holds for $n = 1$. Assume it holds for $n-1$. Then

$$\begin{aligned}
\delta^n([X,Y]) &= \delta\left(\sum_{i=0}^{n-1} \binom{n-1}{i} [\delta^i(X), \delta^{n-1-i}Y]\right) \\
&= \sum_{i=0}^{n-1} \binom{n-1}{i} \delta([\delta^i(X), \delta^{n-1-i}Y]) \\
&= \sum_{i=0}^{n-1} \binom{n-1}{i} [\delta^{i+1}(X), \delta^{n-i-1}Y] + \sum_{i=0}^{n-1} \binom{n-1}{i} [\delta^i(X), \delta^{n-i}Y] \\
&= \sum_{i=1}^{n} \binom{n-1}{i-1} [\delta^i(X), \delta^{n-1-i}Y] + \sum_{i=0}^{n} \binom{n-1}{i} [\delta^i(X), \delta^{n-1-i}Y] \\
&= [\delta^n X, Y] + [X, \delta^n Y] + \sum_{i=1}^{n-1} \left[\binom{n-1}{i-1} + \binom{n-1}{i}\right] [\delta^i(X), \delta^{n-1-i}Y] \\
&= \sum_{i=0}^{n} \binom{n}{i} [\delta^i(X), \delta^{n-i}Y].
\end{aligned}$$

Alternatively we can write

$$\delta^n([X,Y]) = \sum_{\substack{i+j=n \\ i,j\geq 0}} \frac{n!}{i!j!} [\delta^i(X), \delta^j(Y)],$$

so now

$$\begin{aligned}
\exp(\delta)([X,Y]) &= \sum_{n=0}^{\infty} \frac{\delta^n}{n!}([X,Y]) \\
&= \sum_{n=0}^{\infty} \sum_{\substack{i+j=n \\ i,j\geq 0}} \left[\frac{\delta^i}{i!}(X), \frac{\delta^j}{j!}(Y)\right] \\
&= \left[\sum_{i=0}^{\infty} \frac{\delta^i}{i!}(X), \sum_{j=0}^{\infty} \frac{\delta^j}{j!}(X)\right] \\
&= [\exp(\delta)(X), \exp(\delta)(Y)]
\end{aligned}$$

We have shown that every $\delta \in \mathrm{Der}(\mathfrak{g})$ integrates to an element in $\mathrm{Aut}(\mathfrak{g})$ and thus lies in $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$. This concludes the proof of $\mathrm{Der}(\mathfrak{g}) = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$.

Exercise 3: Thm IV.15: Decomposition of the automorphism-group

Now let $(\mathfrak{g}, \theta)$ be a semisimple reduced OSL, which we write as a product $\mathfrak{g} = \bigoplus_{i=1}^{a} \mathfrak{g}_i$ of irreducible OSLs $(\mathfrak{g}_i, \theta_i)$, note that we do not have a Eu-

clidean component since $\mathfrak{g}$ is semisimple. Let $G = \mathrm{Aut}(\mathfrak{g})^\circ$ and $G_i = \mathrm{Aut}(\mathfrak{g}_i)^\circ$. Let $\sigma\colon G \to G, \alpha \mapsto \theta\alpha\theta^{-1}$ and σ_i analogously. Prove that there is a Lie-group-isomorphism $\mathrm{Aut}(\mathfrak{g})^\circ \to \Pi_{i=1}^a(\mathrm{Aut}(\mathfrak{g}_i))^\circ$ which sends the subgroup G^σ to $\Pi_{i=1}^a G_i^{\sigma_i}$

Hint: Use uniqueness of the decomposition of OSLs.

Solution: Let $\alpha \in \mathrm{Aut}(\mathfrak{g})^\circ$. We pick an i and look at $\alpha|_{\mathfrak{g}_i}\colon \mathfrak{g}_i \to \mathfrak{g}$. To be able to view $\alpha|_{\mathfrak{g}_i}$ as an element in $\mathrm{Aut}(\mathfrak{g}_i)^\circ$, we have to show that $\alpha(\mathfrak{g}_i) \subset \mathfrak{g}_i$. Since α is an automorphism, we have

$$\mathfrak{g} = \alpha(\mathfrak{g}) = \alpha\left(\bigoplus_{i=1}^n \mathfrak{g}_i\right) = \bigoplus_{i=1}^n \alpha(\mathfrak{g}_i),$$

which is a decomposition of $\mathfrak{g}$ into irreducible components. By Exercise 1(6), this decomposition is unique up to permutation, so $\alpha(\mathfrak{g}_i) = \mathfrak{g}_j$ for some j. Since α is in the connected component of the identity, we get a path $\gamma\colon [0,1] \to \mathrm{Aut}(\mathfrak{g})$ with $\gamma(0) = \mathrm{Id}_{\mathfrak{g}}$ and $\gamma(1) = \alpha$. By the uniqueness of the decomposition we get for any t a $j(t)$ such that $\gamma(t)(\mathfrak{g}_i) = \mathfrak{g}_{j(t)}$. If we start with $X \in \mathfrak{g}_i \setminus 0$, then $t \mapsto \gamma(t)(X)$ is a continuous path $\in \mathfrak{g}_{j(t)}$, which never reaches 0, since $\gamma(t)$ is an automorphism. Since this path starts in $\mathfrak{g}_i$, it has to stay there and so $j(t) = j(1) = 0$ and $\alpha(\mathfrak{g}_i) = \mathfrak{g}_i$. Therefore $\alpha|_{\mathfrak{g}_i} \in \mathrm{Aut}(\mathfrak{g}_i)^\circ$ and we get the isomorphism

$$\begin{aligned} \mathrm{Aut}(\mathfrak{g})^\circ &\to \prod_{i=1}^a \mathrm{Aut}(\mathfrak{g}_i)^\circ \\ \alpha &\mapsto \sum_{i=1}^a \alpha|_{\mathfrak{g}_i}. \end{aligned}$$

If $\sigma(\alpha) = \alpha$, then it gets mapped to $\alpha|_{\mathfrak{g}_i} = \theta\alpha\theta^{-1}|_{\mathfrak{g}_i} = \theta_i\alpha|_{\mathfrak{g}_i}\theta_i = \sigma_i(\alpha|_{\mathfrak{g}_i})$. So G^σ is sent to G^{σ_i}.

Exercise 4: Thm IV.13: Properties of non-compact type

Show that the map $\varphi\colon \mathfrak{p} \times K \to G, (X,k) \mapsto \exp(X)k$ is regular (the derivative has full rank).

Hint: Recall that

$$\mathrm{D}_X \exp = \mathrm{D}_e\, L_{\exp(X)} \cdot \sum_{n=0}^\infty \frac{(-\operatorname{ad}(X))^n}{(n+1)!}$$

and decompose $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$*, so that you can use*

$$\det\left(\sum_{n=0}^\infty \frac{(\operatorname{ad}(X)^2|_{\mathfrak{p}})^n}{(2n+1)!}\right) \geq 1.$$

Solution: Let $(X,k) \in \mathfrak{p} \times K$. We have

$$\begin{aligned} \varphi\colon \quad & \mathfrak{p} \times K \to G \\ & (X,k) \mapsto \exp(X)k \\ \mathrm{D}_{(X,k)}\,\varphi\colon & T_X\,\mathfrak{p} \times T_k K \to \mathfrak{g} \end{aligned}$$

Let $Y \in T_X \mathfrak{p} \cong \mathfrak{p}$ and $Z \in T_k K \cong \mathfrak{k}$. Then

$$\mathrm{D}_{(X,k)}\, \varphi(Y,Z) = \mathrm{D}_{(X,k)}\, \varphi(Y,0) + \mathrm{D}_{(X,k)}\, \varphi(0,Z).$$

We will now calculate these two derivatives. **1**. Let $\gamma\colon \mathbb{R} \to \mathfrak{p} \times K$ be a smooth path $\gamma(t) = (\gamma_X(t), k)$ with $\gamma_X(0) = X$ and $\dot\gamma(0) = (Y, 0)$. Let $m_G\colon G \times G \to G$ denote the multiplication map with derivative $\mathrm{D}_{(g,h)}\, m_G(Y,Z) = Y + Z$ for $g, h \in G, Y, Z \in \mathfrak{g}$. Then

$$\begin{aligned}
\mathrm{D}_{(X,k)}\, \varphi(Y,0) &= \frac{d}{dt}\bigg|_{t=0} \varphi \circ \gamma(t) \\
&= \frac{d}{dt}\bigg|_{t=0} m_G(\exp(\gamma_X(t)), k) \\
&= \mathrm{D}_{(\exp(X),k)}\, m_G \left(\frac{d}{dt}\bigg|_{t=0} \exp(\gamma_X(t)), \frac{d}{dt}\bigg|_{t=0} k \right) \\
&= \mathrm{D}_{\gamma_X(0)} \exp \dot\gamma_X(t) + 0 \\
&= \mathrm{D}_X \exp Y.
\end{aligned}$$

2. Let $\gamma\colon \mathbb{R} \to \mathfrak{p} \times K$ be a smooth path $\gamma(t) = (X, \gamma_k(t))$ with $\gamma_k(0) = k$ and $\dot\gamma(0) = (0, Z)$. Let $m_G\colon G \times G \to G$ denote the multiplication map with derivative $\mathrm{D}_{(g,h)}\, m_G(Y,Z) = Y + Z$ for $g, h \in G, Y, Z \in \mathfrak{g}$. Then

$$\begin{aligned}
\mathrm{D}_{(X,k)}\, \varphi(0,Z) &= \frac{d}{dt}\bigg|_{t=0} \varphi \circ \gamma(t) \\
&= \frac{d}{dt}\bigg|_{t=0} m_G(\exp(X), \gamma_k(t)) \\
&= \mathrm{D}_{(\exp(X),k)}\, m_G \left(\frac{d}{dt}\bigg|_{t=0} \exp(X), \frac{d}{dt}\bigg|_{t=0} \gamma_k(t) \right) \\
&= 0 + Z.
\end{aligned}$$

For $Y \in \mathfrak{p}, Z \in \mathfrak{k}$, we have $\mathrm{D}_{(X,k)}\, \varphi(Y,Z) = \mathrm{D}_X \exp Y + Z$. This shows that $\mathrm{D}_{(X,k)}\, \varphi$ is surjective on $\mathfrak{k} \subset \mathfrak{g}$. It remains to show that it is surjective on $\mathfrak{p}$. Recall

$$\mathrm{D}_X \exp = \mathrm{D}_e\, L_{\exp(X)} \cdot \sum_{n=0}^{\infty} \frac{(-\operatorname{ad}(X))^n}{(n+1)!},$$

where $\mathrm{D}_e\, L_{\exp(X)}$ is exactly the isomorphism that identifies $T_X \mathfrak{p} \cong \mathfrak{p}$. Note that $\mathrm{D}_X \exp Y$ may take values in both $\mathfrak{k}$ and $\mathfrak{p}$. Since $X, Y \in \mathfrak{p}$, we know that $\operatorname{ad}(X)^n Y \in \mathfrak{p}$ if and only if $n \in 2\mathbb{Z}$. So we can write

$$\sum_{n=0}^{\infty} \frac{(-\operatorname{ad}(X))^n}{(n+1)!} Y = \sum_{n=0}^{\infty} \frac{(\operatorname{ad}(X)^2)^n}{(2n+1)!} Y + Z \in \mathfrak{p} + \mathfrak{k}$$

for some $Z \in \mathfrak{k}$. We know from the proof of Theorem IV.13, that

$$\det \sum_{n=0}^{\infty} \frac{(T_X)^n}{(2n+1)!} \neq 0,$$

so it is invertible. Let $T\colon \mathfrak{p} \to \mathfrak{p}$ be the inverse linear map that satisfies

$$\mathrm{D}_X \exp T(Y) = Y + Z$$

for some $Z \in \mathfrak{k}$.
Now we can finish the proof that $\mathrm{D}_{(X,k)}\,\varphi$ is surjective. Let $Y + Z \in \mathfrak{p} \oplus \mathfrak{k} = \mathfrak{g}$. Let $v = (T(Y), Y - \mathrm{D}_X \exp T(Y) + Z) \in \mathfrak{p} \times \mathfrak{k}$. We have

$$\begin{aligned}
\mathrm{D}_{(X,k)}\,\varphi\; v &= \mathrm{D}_{(X,k)}\,\varphi(T(Y), 0) + \mathrm{D}_{(X,k)}\,\varphi(0, Y - \mathrm{D}_X \exp T(Y) + Z) \\
&= \mathrm{D}_X \exp T(Y) + Y - \mathrm{D}_X \exp T(Y) + Z \\
&= Y + Z.
\end{aligned}$$

Therefore $\mathrm{D}_{(X,k)}\,\varphi$ is surjective and since the dimensions agree also injective, so bijective, i.e. $\mathrm{D}_{(X,k)}\,\varphi$ has full rank and φ is regular.