Solution to Exercise Sheet 5

Exercise 1: Theorem III.22: Decomposition into irreducible parts

Let \((\mathfrak{g}, \theta)\) be a reduced OSL with Cartan decomposition \(\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}\). We can write

\[ \mathfrak{e} = \mathfrak{e}_0 \oplus \bigoplus_{j=1}^{a} \mathfrak{p}_j, \]

where each \(\mathfrak{p}_j\) is \(U\)-invariant and irreducible. We define \(\mathfrak{g}_j = [\mathfrak{p}_j, \mathfrak{p}_j] \oplus \mathfrak{p}_j\).

1. Show that \(\mathfrak{g}_j\) is a \(\theta\)-stable ideal in \(\mathfrak{g}\) for \(1 \leq j \leq a\).

**Solution:** The Cartan-decomposition implies that \(\theta(X) = -X\) for all \(X \in \mathfrak{p}_j \subset \mathfrak{e}\). Similarly \([\mathfrak{p}_j, \mathfrak{p}_j] \subset \mathfrak{u}\) and therefore \(\theta([\mathfrak{p}_j, \mathfrak{p}_j]) = [\mathfrak{p}_j, \mathfrak{p}_j]\) and thus \(\theta(\mathfrak{g}_j) = \mathfrak{g}_j\).

**Claim 1:** \([\mathfrak{u}, \mathfrak{g}_j] \subset \mathfrak{g}_j\).

**Proof:** \(\mathfrak{p}_j\) is \(U\)-invariant, which means that \([\mathfrak{u}, \mathfrak{p}_j] \subset \mathfrak{p}_j\). Together with the Jacobi-identity this results in the required inclusion:

\[ [\mathfrak{u}, \mathfrak{g}_j] = [\mathfrak{u}, [\mathfrak{p}_j, \mathfrak{p}_j] + \mathfrak{p}_j] = [\mathfrak{u}, [\mathfrak{p}_j, \mathfrak{p}_j]] + [\mathfrak{u}, \mathfrak{p}_j] = -[\mathfrak{p}_j, [\mathfrak{u}, \mathfrak{p}_j]] - [\mathfrak{p}_j, [\mathfrak{u}, \mathfrak{p}_j]] + \mathfrak{p}_j \subset [\mathfrak{p}_j, \mathfrak{p}_j] + \mathfrak{p}_j = \mathfrak{g}_j \]

**Claim 2:** \([\mathfrak{p}_j, \mathfrak{p}_i] \oplus \mathfrak{e}_0\) = 0 for \(i \neq j\).

**Proof:** Let \(X \in \mathfrak{p}_j\) and \(Y \in \mathfrak{p}_i \oplus \mathfrak{e}_0\). We note that \([X, Y] \in [\mathfrak{e}, \mathfrak{e}] \subset \mathfrak{u}\). We can take a look at the function \(\mathfrak{u} \rightarrow \mathbb{R}\), defined by

\[ Z \mapsto B_\mathfrak{g}([X, Y], Z) = -B_\mathfrak{g}(Y, [X, Z]) = -\langle AY, [X, Z] \rangle \in -(\mathfrak{p}_j \oplus \mathfrak{e}_0, \mathfrak{p}_j) = 0, \]

where we used \(\mathfrak{ad} \mathfrak{g}\)-invariance of \(B_\mathfrak{g}\). Claim 1 and the orthogonality of the decomposition \(\mathfrak{e} = \mathfrak{e}_0 \oplus \bigoplus_{i=1}^{a} \mathfrak{p}_i\). Now \(B_\mathfrak{g}|\mathfrak{u} \oplus \mathfrak{u} < 0\) is negative definite (Lemma III.6) and \(Z \mapsto B_\mathfrak{g}([X, Y], Z)\) is the zero map, which means that \([X, Y] = 0\).

Now we see that

\[ [\mathfrak{g}, \mathfrak{g}_j] = [\mathfrak{u}, \mathfrak{g}_j] + [\mathfrak{e}_0 \oplus \bigoplus_{i=j}^{a} \mathfrak{p}_i, \mathfrak{g}_j] \]

\[ \subset \mathfrak{g}_j + [\mathfrak{e}_0 \oplus \bigoplus_{i=j}^{a} \mathfrak{p}_i, [\mathfrak{p}_j, \mathfrak{p}_j] + \mathfrak{p}_j] = [\mathfrak{p}_j, [\mathfrak{p}_j, \mathfrak{p}_j] + \mathfrak{p}_j] \quad \text{(Claim 1)} \]

\[ = \mathfrak{g}_j + [\mathfrak{e}_0 \oplus \bigoplus_{i=j}^{a} \mathfrak{p}_i, [\mathfrak{p}_j, \mathfrak{p}_j]] + [\mathfrak{p}_j, [\mathfrak{p}_j, \mathfrak{p}_j]] + [\mathfrak{p}_j, \mathfrak{p}_j] \quad \text{(Claim 2)} \]

\[ \subset \mathfrak{g}_j - [\mathfrak{p}_j, [\mathfrak{p}_j, \mathfrak{e}_0 \oplus \bigoplus_{i=j}^{a} \mathfrak{p}_i]] - [\mathfrak{p}_j, [\mathfrak{e}_0 \oplus \bigoplus_{i=j}^{a} \mathfrak{p}_i]] + [\mathfrak{p}_j, \mathfrak{u}] \quad \text{(Jacobi, } [\mathfrak{e}, \mathfrak{e}] \subset \mathfrak{u}) \]

\[ \subset \mathfrak{g}_j - [\mathfrak{p}_j, \mathfrak{u}] - [\mathfrak{p}_j, \mathfrak{u}] + \mathfrak{g}_j \subset \mathfrak{g}_j \quad \text{(Claim 1)} \]

which concludes the proof that \(\mathfrak{g}_j\) is a \(\theta\)-stable ideal in \(\mathfrak{g}\).
(2) From (1) and Sheet 4, exercise 2(4) it follows that $B_{g_j} = B_g|_{g_j \times g_j}$. Show that $B_{g_j}$ is non-degenerate, i.e. $g_j$ is semi-simple.

Solution: We have to show that if $X \in g_j$ satisfies $B_{g_j} (X, g_j) = 0$, then $X = 0$. Since $B_{g_j}$ is bilinear, we can check it in two cases:

**Case 1:** Let $X \in p_j \subset g_j$ with $B_{g_j} (X, g_j) = 0$. In particular $0 = B_{g_j} (X, X) = \langle AX, X \rangle = c_j \cdot \langle X, X \rangle$ with $c_j \neq 0$, so $X = 0$.

**Case 2:** Let $X \in [p_j, p_j]$ with $B_{g_j} (X, g_j) = 0$. We decompose $u = [p_j, p_j] \oplus u'$ orthogonally with respect to $B_{g|u \times u} << 0$. Now we have

$$B_{g}(X,u) = B_{g}(X,[p_j, p_j]) + B_{g}(X, u') = 0 + 0,$$

and since $B_{g|u \times u} << 0$, we get that $X = 0$.

(3) Let

$$m = \bigoplus_{j=1}^{a} g_j.$$

Prove that the centralizer

$$g_0 := Z_{g} (m) = \{ X \in g : [X, m] = 0 \ \forall m \in m \}$$

is a $\theta$-stable ideal in $g$.

Solution. By (1), $g_j$ (and direct products thereof) are $\theta$-stable, so let $m \in m$ and $X \in g_0$. Then $[\theta(X), m] = \theta ([X, \theta (m)]) = \theta ([X, m]) = \theta (0) = 0$. So $g_0$ is $\theta$-stable.

To see that $g_0$ is an ideal in $g$ let $m \in m$,

$$[[g_0, g], m] = -[[g, m], g_0] - [[m, g_0], g]$$

$$\subset -[m, g_0] - [0, g] = 0$$

(4) Show that this gives rise to an orthogonal decomposition $g = g_0 \oplus m$.

Solution: We need to show that the orthogonal complement of $m$ is $g_0$. We claim that the orthogonal complement

$$m^\perp := \{ X \in g : B_g (X, m) = 0 \}$$

is an ideal in $g$. Indeed:

$$B_g([g, m^\perp], m) = -B_g(m^\perp, [g, m]) \subset -B_g(m^\perp, m) = 0.$$

Both $m$ and $m^\perp$ are ideals in $g$, so $[m^\perp, m] \subset m \cap m^\perp = 0$, so $m^\perp \subset g_0$.

On the other hand, the map $\text{ad}(g_0) \circ \text{ad}(m)$ sends $X$ to

$$[g_0, [m, X]] = -[m, [X, g_0]] - [X, [g_0, m]]$$

$$\subset -[m, g_0] - [X, g_0 \cap m]$$

$$\subset -g_0 \cap m - [X, 0] = 0$$

since both $g_0$ and $m$ are ideals (by (3) and (1)) in $g$ and $g_0 \cap m = 0$ by definition of $g_0$. Therefore $B_g(g_0, m) = \text{tr}(\text{ad}(g_0) \circ \text{ad}(m)) = 0$, so $g_0 \subset m^\perp$. This concludes the proof that $g_0 = m^\perp$. 

2
(5) Show that \((\mathfrak{g}_0, \theta|_{\mathfrak{g}_0})\) is a reduced OSL of Euclidean type (or 0).

Consider \(u_0 = \{ X \in \mathfrak{g}_0 : \theta(X) = X \}\). By (4), the effective OSL is an orthogonal product \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{m}\) with \(u \subset \mathfrak{g}\) compactly embedded. By Sheet 4, exercise 2(6), also \(u_0 \subset \mathfrak{g}\) is compactly embedded. If \(\mathfrak{g}_0 \neq 0\), then \(\theta\) is an involution and \(\mathfrak{g}_0\) is an OSL.

Assume that \(i \subset u_0\) is an ideal in \(\mathfrak{g}_0\). Then \([i, \mathfrak{g}] = [i, \mathfrak{g}_0] + \sum_j [i, \mathfrak{g}_j] \subset i + 0\), so \(i \subset u_0 \subset u\) is also an ideal in \(\mathfrak{g}\). Since \((\mathfrak{g}, \theta)\) is reduced, \(u\) contains no non-trivial ideals in \(\mathfrak{g}\), so \(i\) is trivial. This means that \(\mathfrak{g}_0\) is reduced.

Let \(X \in u\). Then \(B_\theta([\mathfrak{e}_0, \mathfrak{e}_0], [\mathfrak{e}_0, \mathfrak{e}_0]) = -B_\theta(\mathfrak{e}_0, [\mathfrak{e}_0, \mathfrak{e}_0]) = (A \mathfrak{c}_0, [\mathfrak{e}_0, \mathfrak{e}_0]) = 0\) (by the definition of \(\mathfrak{e}_0\)). Since \(B_\theta|_{u \times u} <= 0\), we get \([\mathfrak{e}_0, \mathfrak{e}_0] = 0\). Now \([\mathfrak{c}_0, \mathfrak{c}_0] = [\mathfrak{c}_0, \mathfrak{c}_0] + [\mathfrak{e}_0, \mathfrak{e}_0] \subset 0 + u_0\) implies that \(\mathfrak{c}_0\) is an abelian ideal in \(\mathfrak{g}_0\), i.e \((\mathfrak{g}_0, \theta|_{\mathfrak{g}_0})\) is of Euclidean type.

(6) Show that the decomposition \(\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{j=1}^\alpha \mathfrak{g}_j = \mathfrak{g}_0' \oplus \bigoplus_{k=1}^{\alpha'} \mathfrak{g}_k'\) (orthogonal wrt \(B_\theta\)), where the \(\mathfrak{g}_j\) and \(\mathfrak{g}_k'\) are irreducible ideals in \(\mathfrak{g}\). Define \(\pi : \mathfrak{g} \to \mathfrak{g}_0\) to be the orthogonal projection to \(\mathfrak{g}_0\).

**Lemma 1:** If \(\mathfrak{n} \subset \mathfrak{g}\) is a semi-simple ideal in \(\mathfrak{g}\), then \(\pi(\mathfrak{n})\) is also a semi-simple ideal.

**Proof:** For \(X \in \mathfrak{n}\), we can write uniquely \(X = \pi(X) + Y\) for \(Y \in \mathfrak{m} := \bigoplus_{i=1}^\alpha \mathfrak{g}_i\).

Let \(Z \in \mathfrak{g}\), then \([X, Z] = [\pi(X) + Y, Z] = [\pi(X), Z] + [Y, Z] \) is the decomposition into \(\mathfrak{g}_0\) and \(\mathfrak{m}\), because \([\pi(X), Z] \in \mathfrak{g}_0\) and \([Y, Z] \in \mathfrak{m}\). This implies \([\pi(X), Z] = \pi([X, Z]) \in \pi(\mathfrak{n})\), since \(\mathfrak{n}\) is an ideal in \(\mathfrak{g}\). This shows that \(\pi(\mathfrak{n})\) is an ideal.

Let \(\pi(X) \in \pi(\mathfrak{n})\). Assume that for every \(Y \in \mathfrak{g}\), \(B_\theta(\pi(x), \pi(Y)) = 0\). Note that we used \(B_\theta = B_\theta|_{\mathfrak{n} \times \mathfrak{n}}\) since \(\mathfrak{n}\) is an ideal in \(\mathfrak{g}\) (see sheet 4, exercise 2(4)). We can write \(Y = \pi(Y) + Y'\), where \(\pi(Y)\) and \(Y'\) are orthogonal wrt \(B_\theta\), in fact \(Y'\) is orthogonal to all of \(\pi(n)\), in particular \(B(\pi(X), Y') = 0\). So we have that for all \(Y \in \mathfrak{n}\),

\[B_\theta(\pi(X), Y) = B_\theta(\pi(X), \pi(Y)) + B_\theta(\pi(X), Y') = 0.\]

Since \(\mathfrak{n}\) is semi-simple, \(\pi(X) = 0\). This concludes the proof that \(\pi(\mathfrak{n})\) is semi-simple.

Now we apply Lemma 1 to the semi-simple ideal \(\mathfrak{m}' = \bigoplus_{i=1}^{\alpha'} \mathfrak{g}_i'\) of \(\mathfrak{g}\) to get that \(\pi(\mathfrak{m}') \subset \mathfrak{g}_0\) is a semi-simple ideal in \(\mathfrak{g}\). Since \(\mathfrak{g}_0\) is of Euclidean type, we have that \(\mathfrak{p}_0 := \mathfrak{g}_0 \cap \mathfrak{p}\) is an abelian ideal in \(\mathfrak{g}_0\). So semi-simplicity of \(\pi(\mathfrak{m}')\) implies \(\pi(\mathfrak{m}') \cap \mathfrak{p}_0 = 0\). Thus also \(\pi(\mathfrak{m}')_p = 0\). So \(\pi(\mathfrak{m}') \in \mathfrak{g}_0(\mathfrak{p}_0) = \mathfrak{p}_0\) and thus \(\pi(\mathfrak{m}') = 0\). We have shown that \(\mathfrak{m}' \subset \mathfrak{g}_0 = \mathfrak{g}\). Doing the same with the projection \(\mathfrak{g} \to \mathfrak{g}_0\), results in \(\mathfrak{m} \subset \mathfrak{m}'\), so \(\mathfrak{m} = \mathfrak{m}'\) and \(\mathfrak{g}_0 = \mathfrak{g}'_0\).
Now we have to show that the semi-simple part of the decomposition is unique (up to permutation). We define \( p_i = g_i \cap p \) and \( p_i' = g_i' \cap p \) for \( i \in \{1, \ldots, a\} \) and \( k \in \{1, \ldots, a'\} \). We define \( \pi_i: g \to p_i \). We note that \( \text{ad}(t) \) and the \( \pi_i \) commute:

**Lemma 2:** For all \( X \in \mathfrak{f}, Y \in p \), we have \( \pi_i([X,Y]) = [X,\pi_i(Y)] \).

**Proof:** We can write \( Y = \sum_{i=0}^{a} Y_i \) uniquely, where \( Y_i = \pi_i(Y) \in p_i \). Then we can also write

\[
[X,Y] = \left[ X, \sum_{i=0}^{a} Y_i \right] = \sum_{j=0}^{a} [X,Y_i]
\]

uniquely, where \([X,Y_i] \in g_i \cap p = p_j\) because \( g_j \) is an ideal and \([\mathfrak{f},p] \subset p \). Therefore \( \pi_i([X,Y]) = [X,Y_i] \in [X,\pi_i(Y)] \).

Now look at a \( p_i' \) and its image \( \pi_i(p_i') \subset p_i \). Note that \( g_i' \) and thus \( p_i' \) are \( \text{ad}(t) \)-invariant. By Lemma 2, \( \text{ad}(t)(\pi_i(p_i')) = \pi_i(\text{ad}(t)(p_i')) \subset \pi_i(p_i') \), i.e. \( \pi_i(p_i') \) is \( \text{ad}(t) \)-invariant. Since \( p_i \) is irreducible, the only invariant subspaces are 0 and \( p_i \), so we \( \pi_i(p_i') = 0 \) or \( \pi_i(p_i') = p_i \). We know that \( p = \bigoplus p_i = \bigoplus p_i' \), so for every \( p_i' \), there exists an \( i \) such that \( \pi_i(p_i') = p_i \). To get a bijection between the factors, we have to make sure that for any \( k \) there is at most one such \( i \).

**Lemma 3:** If \( \pi_i(p_i') = p_i \), then there is a vectorspace-isomorphism \( p_k' \to p_j \) given by \( \pi_i \).

**Proof:** The map \( \pi_i|_{p_k'}: p_k' \to p_j \) is surjective by assumption. Let \( Y \in \ker(\pi_i|_{p_k'}) \), i.e. \( Y \in p_k' \) with \( \pi_i(Y) = 0 \). By Lemma 2, \( \pi_i([\mathfrak{f},Y]) = [\mathfrak{f},\pi_i(Y)] = 0 \), which means that the kernel is \( \text{ad}(t) \)-invariant. Since \( p_k' \) is irreducible, the kernel has to be 0 or all of \( p_k' \). Since \( p_i \neq 0 \) and \( \pi_i|_{p_k'} \) is surjective, the kernel cannot be all of \( p_k' \); so it has to be 0. We conclude that \( \pi_i|_{p_k'} \) is an isomorphism.

Assume there are \( i,j \) with \( \pi_i(p_i') = p_i \) and \( \pi_j(p_i') = p_j \). Then using Lemma 3, we get a vectorspace-isomorphism \( \varphi: p_i \to p_j \) which to \( Y_k \in p_k' \) with \( \pi_j(Y_k) \). In particular \( \pi_j = \varphi \circ \pi_i \). Considering Lemma 2, we get

\[
\text{ad}(t) \circ \varphi(Y_k) = [\mathfrak{f},\pi_j(Y_k)] = [\mathfrak{f},\pi_j(Y_k)] = \varphi \circ \pi_i([\mathfrak{f},Y_k])
\]

To show that \( i = j \), we have to study the \( \text{ad}(t) \)-action on \( p \) in more detail:

**Lemma 4:** For a fixed \( i \), we have that the kernel of the \( \text{ad}(t_i) \)-action on \( p \) is

\[
\ker(\text{ad}(t_i)) = \{ Y \in p : [t_i,Y] = 0 \} = \bigoplus_{j \neq i} p_j.
\]

**Proof:** We first claim that \( \text{ad}(t_i) \) acts non-trivially on \( p_i \); in fact, for all \( X \in \mathfrak{t}_i \), \( \text{ad}(X): p_i \to p_i \) is \( \equiv 0 \) if and only if \( X = 0 \). The direction \( X = 0 \Rightarrow \text{ad}(X) \equiv 0 \) is immediate. For \( "\Leftarrow" \), let \( \text{ad}(X)|_{p_i} \equiv 0 \). For all \( Y \in p_i \), we get the relations \( [X,Y] = 0, [X,t_i] \subset \mathfrak{t}_i, [Y,p_i] \subset \mathfrak{t}_i, [Y,t_i] \subset p_i \), which allow us to write as matrices

\[
\text{ad}(X) = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \quad \text{and} \quad \text{ad}(Y) = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix},
\]

where \( * \) stands for any matrix, and thus \( \equiv 0 \).
where we decomposed \( g_i = p_j \oplus t_i \). Now

\[
B_g(X,Y) = \text{tr}(\text{ad}(X) \text{ad}(Y)) = \text{tr} \left( \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right) = \text{tr} \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} = 0
\]

for all \( Y \in p_i \) and since \( g_i \) is an ideal, \( B_{g_i} = B_g|_{g_i \times g_i} \), so we can use semi-simplicity of \( g_i \) to see that \( X = 0 \).

Now we know that \( \text{ad}(t_i) \) is non-zero on \( p_j \). Note that for \( i \neq j \), we have \( \text{ad}(t_i)(p_j) = [t_i, p_j] \subset [g_i, g_j] \subset g_i \cap g_j = 0 \). This shows the stated form of the kernel of the \( \text{ad}(t_i) \)-action on \( p \).

Assume there are \( i \neq j \), such that \( \pi_i(p_k) = p_i \) and \( \pi_j(p_k) = p_j \). By Lemma 4, \( \text{ad}(t_i)|_{p_j} \equiv 0 \). So let \( Y_i \in p_i \setminus \{ 0 \} \) with \( \text{ad}(t_i)(Y_i) \neq 0 \). Then

\[
0 \neq \varphi \circ \text{ad}(t_i)(Y_i) = \text{ad}(t_i) \circ \varphi(Y_i) = 0
\]

results in a contradiction. (Essentially we have used that even though two different \( p_i, p_j \) may be isomorphic as vector spaces, when you view them as \( \text{ad}(t) \)-modules, they are no longer isomorphic.)

This means that to any \( p_k \), exactly one space \( p_i = p_j \) is associated. We conclude that \( \alpha = \alpha' \) and \( \{ p_i \}_{i=1}^a = \{ p_j \}_{k=1}^a \).

Finally, since \( g_i = p_i \oplus t_i \) and \( g_k = p_k' \oplus t_k' \), we also get that \( \{ g_i \}_{i=1}^a = \{ g_k' \}_{k=1}^a \).

We have shown that the factors are unique up to reordering.

**Exercise 2: Lemma IV.10: Automorphism group**

Let \( g \) be a real Lie-algebra. Prove that \( \text{Aut}(g) \) is a Lie-subgroup of \( \text{GL}(g) \) with Lie-algebra \( \text{Der}(g) \).

**Hint: The Lie algebra of a Lie-subgroup \( H < G \) is**

\[
\text{Lie}(H) = \{ X \in \text{Lie}(G) : \forall t \in \mathbb{R} : \exp(tX) \in H \}.
\]

**Solution:** If we choose a basis \( \{ e_1, \ldots, e_n \} \) for \( g \), we can identify \( \text{GL}(g) = \text{GL}(n) \).

Then by linearity

\[
\text{Aut}(g) = \{ g \in \text{GL}(g) : \forall X, Y \in g : g([X,Y]) = [g(X),g(Y)] \}
\]

\[
= \{ g \in \text{GL}(n) : \forall i,j : g([e_i,e_j]) = [g(e_i),g(e_j)] \}
\]

which is a closed subgroup of \( \text{GL}(n) \). Closed subgroups of Lie-groups are Lie-groups, so \( \text{Aut}(g) \) is a Lie-subgroup of \( \text{GL}(g) \).

Let \( \delta \in \text{Lie}(\text{Aut}(g)) \). Then \( \exp(t\delta) \in \text{Aut}(g) \), so for all \( X, Y \in g \),

\[
\exp(t\delta)([X,Y]) = [\exp(t\delta)(X), \exp(t\delta)(Y)].
\]

(1)

Since we are working with matrix groups, we can write \( \exp(t\delta) = \text{Id} + t\delta + \frac{1}{2} t^2 \delta^2 + \ldots \) and taking the derivative results in

\[
\delta([X,Y]) = \frac{d}{dt} \bigg|_{t=0} [X,Y] + t\delta([X,Y]) + \frac{1}{2} t^2 \delta^2([X,Y]) + \ldots
\]

\[
= \frac{d}{dt} \bigg|_{t=0} [X + t\delta(X) + \frac{1}{2} t^2 \delta^2(X) + \ldots, Y + t\delta(Y) + \frac{1}{2} t^2 \delta^2(Y) + \ldots]
\]

\[
= \frac{d}{dt} \bigg|_{t=0} [X, Y] + t \cdot ([\delta(X), Y] + [X, \delta(Y)]) + t^2 \cdot (\ldots) + \ldots
\]

\[
= [\delta(X), Y] + [X, \delta(Y)].
\]
So if $\delta \in \text{Lie}(\text{Aut}(g))$ then $\delta \in \text{Der}(g)$.

Now let $\delta \in \text{Der}(g)$. We need to show that $\exp(\delta) \in \text{Aut}(g)$. To use the definition of $\exp$ for matrix-groups, we first need to understand $\delta^n([X,Y])$. We prove by induction

$$\delta^n([X,Y]) = \sum_{i=0}^{n} \binom{n}{i} [\delta^i(X), \delta^{n-i}Y].$$

The formula holds for $n = 1$. Assume it holds for $n - 1$. Then

$$\begin{align*}
\delta^n([X,Y]) &= \delta \left( \sum_{i=0}^{n-1} \binom{n-1}{i} [\delta^i(X), \delta^{n-1-i}Y] \right) \\
&= \sum_{i=0}^{n-1} \binom{n-1}{i} \delta([\delta^i(X), \delta^{n-1-i}Y]) \\
&= \sum_{i=0}^{n-1} \binom{n-1}{i} [\delta^{i+1}(X), \delta^{n-i-1}Y] + \sum_{i=0}^{n-1} \binom{n-1}{i} [\delta^i(X), \delta^{n-i}Y] \\
&= \sum_{i=1}^{n} \binom{n-1}{i-1} [\delta^i(X), \delta^{n-i}Y] + \sum_{i=0}^{n-1} \binom{n-1}{i} [\delta^i(X), \delta^{n-i}Y] \\
&= [\delta^nX, Y] + [X, \delta^nY] + \sum_{i=1}^{n-1} \left[ \binom{n-1}{i-1} + \binom{n-1}{i} \right] [\delta^i(X), \delta^{n-1-i}Y] \\
&= \sum_{i=0}^{n} \binom{n}{i} [\delta^i(X), \delta^{n-i}Y].
\end{align*}$$

Alternatively we can write

$$\delta^n([X,Y]) = \sum_{i+j=n, i,j \geq 0} \frac{n!}{i!j!} [\delta^i(X), \delta^j(Y)],$$

so now

$$\begin{align*}
\exp(\delta)([X,Y]) &= \sum_{n=0}^{\infty} \frac{\delta^n}{n!}([X,Y]) \\
&= \sum_{n=0}^{\infty} \sum_{i+j=n, i,j \geq 0} \frac{\delta^i}{i!} \frac{\delta^j}{j!} \\
&= \left[ \sum_{i=0}^{\infty} \frac{\delta^i}{i!} (X), \sum_{j=0}^{\infty} \frac{\delta^j}{j!} (X) \right] \\
&= [\exp(\delta)(X), \exp(\delta)(Y)].
\end{align*}$$

We have shown that every $\delta \in \text{Der}(g)$ integrates to an element in $\text{Aut}(g)$ and thus lies in $\text{Lie}(\text{Aut}(g))$. This concludes the proof of $\text{Der}(g) = \text{Lie}(\text{Aut}(g))$.

**Exercise 3: Thm IV.15: Decomposition of the automorphism-group**

Now let $(g, \theta)$ be a semisimple reduced OSL, which we write as a product $g = \bigoplus_{i=1}^{a} g_i$ of irreducible OSLs $(g_i, \theta_i)$, note that we do not have a Eu-
clidean component since \( g \) is semisimple. Let \( G = \text{Aut}(g)^\circ \) and \( G_i = \text{Aut}(g_i)^\circ \).
Let \( \sigma: G \to G, \alpha \mapsto \theta \alpha \theta^{-1} \) and \( \sigma_i \) analogously. Prove that there is a Lie-group-isomorphism \( \text{Aut}(g)^\circ \to \prod_{i=1}^\alpha (\text{Aut}(g_i))^\circ \) which sends the subgroup \( G^\sigma \) to \( \prod_{i=1}^\alpha G_i^\sigma \).

Hint: Use uniqueness of the decomposition of OSLs.

Solution: Let \( \alpha \in \text{Aut}(g)^\circ \). We pick an \( i \) and look at \( \alpha|_{g_i}: g_i \to g_i \). To be able to view \( \alpha|_{g_i} \) as an element in \( \text{Aut}(g_i)^\circ \), we have to show that \( \alpha(g_i) \subset g_i \). Since \( \alpha \) is an automorphism, we have

\[
\alpha(g) = \alpha \left( \bigoplus_{i=1}^n g_i \right) = \bigoplus_{i=1}^n \alpha(g_i),
\]

which is a decomposition of \( g \) into irreducible components. By Exercise 1(6), this decomposition is unique up to permutation, so \( \alpha(g_i) = g_j \) for some \( j \). By the uniqueness of the decomposition we get for any \( t \in [0, 1] \) such that \( \gamma(t)(g_i) = g_j(t) \). If we start with \( X \in g_i \setminus 0 \), then \( \gamma(t)(X) \) is a continuous path in \( g_j(t) \), which never reaches 0, since \( \gamma(t) \) is an automorphism. Since this path starts in \( g_i \), it has to stay there and so \( j(t) = j(1) = 0 \) and \( \alpha(g_i) = g_i \). Therefore \( \alpha|_{g_i} \in \text{Aut}(g_i)^\circ \) and we get the isomorphism

\[
\text{Aut}(g)^\circ \to \prod_{i=1}^\alpha \text{Aut}(g_i)^\circ
\]

\[
\alpha \mapsto \sum_{i=1}^\alpha \alpha|_{g_i}.
\]

If \( \sigma(\alpha) = \alpha \), then it gets mapped to \( \alpha|_{g_i} = \theta \alpha \theta^{-1}|_{g_i} = \theta_i \alpha|_{g_i} \theta_i = \sigma_i(\alpha|_{g_i}) \). So \( G^\sigma \) is sent to \( G^\sigma_i \).

Exercise 4: Thm IV.13: Properties of non-compact type

Show that the map \( \varphi: \mathfrak{p} \times K \to G, (X, k) \mapsto \exp(X)k \) is regular (the derivative has full rank).

Hint: Recall that

\[
\text{D}_X \exp = D_x \exp(X) \cdot \sum_{n=0}^\infty \frac{(-\text{ad}(X))^n}{(n+1)!}
\]

and decompose \( g = \mathfrak{k} + \mathfrak{p} \), so that you can use

\[
\det \left( \sum_{n=0}^\infty \frac{\text{ad}(X)^2|_\mathfrak{p}}{(2n+1)!} \right) \geq 1.
\]

Solution: Let \( (X, k) \in \mathfrak{p} \times K \). We have

\[
\varphi: \mathfrak{p} \times K \to G
\]

\[
(X, k) \mapsto \exp(X)k
\]

\[
\text{D}_{(X, k)} \varphi: \mathfrak{p} \times T_k K \to g
\]
Let \( Y \in T_X \mathfrak{p} \cong \mathfrak{p} \) and \( Z \in T_k K \cong \mathfrak{k} \). Then

\[
D_{(X,k)} \varphi(Y, Z) = D_{(X,k)} \varphi(Y, 0) + D_{(X,k)} \varphi(0, Z).
\]

We will now calculate these two derivatives. 1. Let \( \gamma : \mathbb{R} \to \mathfrak{p} \times K \) be a smooth path \( \gamma(t) = (\gamma_X(t), k) \) with \( \gamma_X(0) = X \) and \( \dot{\gamma}(0) = (Y, Z) \). Let \( m_G : G \times G \to G \) denote the multiplication map with derivative \( D_{(g,h)} m_G(Y, Z) = Y + Z \) for \( g, h \in G, Y, Z \in \mathfrak{g} \). Then

\[
D_{(X,k)} \varphi(Y, 0) = \left. \frac{d}{dt} \right|_{t=0} \varphi \circ \gamma(t)
= \left. \frac{d}{dt} \right|_{t=0} m_G(\exp(\gamma_X(t)), k)
= D_{(\exp(X), k)} m_G \left( \left. \frac{d}{dt} \right|_{t=0} \exp(\gamma_X(t)), \left. \frac{d}{dt} \right|_{t=0} k \right)
= D_{\gamma_X(0)} \exp(\dot{\gamma}_X(t)) = 0 + Y.
\]

2. Let \( \gamma : \mathbb{R} \to \mathfrak{p} \times K \) be a smooth path \( \gamma(t) = (X, \gamma_k(t)) \) with \( \gamma_k(0) = k \) and \( \dot{\gamma}(0) = (0, Z) \). Let \( m_G : G \times G \to G \) denote the multiplication map with derivative \( D_{(g,h)} m_G(Y, Z) = Y + Z \) for \( g, h \in G, Y, Z \in \mathfrak{g} \). Then

\[
D_{(X,k)} \varphi(0, Z) = \left. \frac{d}{dt} \right|_{t=0} \varphi \circ \gamma(t)
= \left. \frac{d}{dt} \right|_{t=0} m_G(\exp(X), \gamma_k(t))
= D_{(\exp(X), k)} m_G \left( \left. \frac{d}{dt} \right|_{t=0} \exp(X), \left. \frac{d}{dt} \right|_{t=0} \gamma_k(t) \right)
= 0 + Z.
\]

For \( Y \in \mathfrak{p}, Z \in \mathfrak{k} \), we have \( D_{(X,k)} \varphi(Y, Z) = D_X \exp Y + Z \). This shows that \( D_{(X,k)} \varphi \) is surjective on \( \mathfrak{k} \subset \mathfrak{g} \). It remains to show that it is surjective on \( \mathfrak{p} \). Recall

\[
D_X \exp = D_e \cdot \exp(X) \cdot \sum_{n=0}^{\infty} \frac{(-\operatorname{ad}(X))^n}{(n+1)!},
\]

where \( D_e \cdot \exp(X) \) is exactly the isomorphism that identifies \( T_X \mathfrak{p} \cong \mathfrak{p} \). Note that \( D_X \exp Y \) may take values in both \( \mathfrak{k} \) and \( \mathfrak{p} \). Since \( X, Y \in \mathfrak{p} \), we know that \( \operatorname{ad}(X)^n Y \in \mathfrak{p} \) if and only if \( n \in 2\mathbb{Z} \). So we can write

\[
\sum_{n=0}^{\infty} \frac{(-\operatorname{ad}(X))^n}{(n+1)!} Y = \sum_{n=0}^{\infty} \frac{(\operatorname{ad}(X)^2)^n}{(2n+1)!} Y + Z \in \mathfrak{p} + \mathfrak{k}
\]

for some \( Z \in \mathfrak{k} \). We know from the proof of Theorem IV.13, that

\[
\det \sum_{n=0}^{\infty} \frac{(T_X)^n}{(2n+1)!} \neq 0,
\]

so it is invertible. Let \( T : \mathfrak{p} \to \mathfrak{p} \) be the inverse linear map that satisfies

\[
D_X \exp T(Y) = Y + Z
\]
for some $Z \in \mathfrak{k}$.

Now we can finish the proof that $D_{(X,k)} \varphi$ is surjective. Let $Y + Z \in p \oplus \mathfrak{k} = g$.

Let $v = (T(Y), Y - D_X \exp T(Y) + Z) \in p \times \mathfrak{k}$. We have

$$
D_{(X,k)} \varphi \; v = D_{(X,k)} \varphi(T(Y), 0) + D_{(X,k)} \varphi(0, Y - D_X \exp T(Y) + Z)
$$

$$
= D_X \exp T(Y) + Y - D_X \exp T(Y) + Z
$$

$$
= Y + Z.
$$

Therefore $D_{(X,k)} \varphi$ is surjective and since the dimensions agree also injective, so bijective, i.e. $D_{(X,k)} \varphi$ has full rank and $\varphi$ is regular.