Solution to Exercise Sheet 6

Exercise 1: Duality between $\mathbb{S}^n$ and $\mathbb{H}^n$

Consider the Lie algebra

$$\mathfrak{so}(n) = \left\{ X \in \mathfrak{gl}(n, \mathbb{R}) \colon {}^tX + X = 0 \right\}$$

and for $p + q = n$, define

$$\begin{array}{rcl} \theta_{p,q} \colon \mathfrak{gl}(n,\mathbb{R}) & \to & \mathfrak{gl}(n,\mathbb{R}) \\ X & \mapsto & I_{p,q} X I_{p,q} \end{array} \quad \text{where} \quad I_{p,q} = \begin{pmatrix} -\operatorname{Id}_p & 0 \\ 0 & \operatorname{Id}_q \end{pmatrix}$$

(1) Show that for any $p + q = n$, $(\mathfrak{so}(n), \theta_{p,q})$ is an orthogonal symmetric Lie algebra

Solution: We see that $\theta_{p,q}$ sends $\mathfrak{so}(n)$ to itself, since

$${}^t\theta_{p,q}(X) + \theta_{p,q}(X) I_{p,q} {}^tX I_{p,q} + I_{p,q} X I_{p,q} = I_{p,q}({}^tX + X) I_{p,q} = 0$$

for all $X \in \mathfrak{so}(n)$. For $p, q \geq 1$, $\theta_{p,q}$ is not the identity, since

$$\theta_{p,q}\left(\begin{array}{cc|cc} \cos(t) & 0 & -\sin(t) & 0 \\ 0 & 0_{p-1} & 0 & 0 \\ \hline \sin(t) & 0 & \cos(t) & 0 \\ 0 & 0 & 0 & 0_{q-1} \end{array} \right) = \left(\begin{array}{cc|cc} \cos(t) & 0 & \sin(t) & 0 \\ 0 & 0_{p-1} & 0 & 0 \\ \hline -\sin(t) & 0 & \cos(t) & 0 \\ 0 & 0 & 0 & 0_{q-1} \end{array} \right)$$

and clearly $\theta_{p,q}^2 = \operatorname{Id}$.
For

$$X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \in \mathfrak{so}(n), \quad \text{we have} \quad \theta_{p,q}(X) = \left(\begin{array}{c|c} A & -B \\ \hline -C & D \end{array} \right),$$

so the eigenspace of $\theta_{p,q}$ associated to $+1$ is

$$\mathfrak{k} = \mathfrak{s}(\mathfrak{o}(p) \times \mathfrak{o}(q)) = \{X \in \mathfrak{so}(n) \colon I_{p,q} X I_{p,q} = X\}$$

which integrates to the closed subgroup

$$\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q)) = \{g \in \mathrm{SO}(n) \colon I_{p,q} g I_{p,q} = g\} < \mathrm{SO}(n),$$

so since $\mathfrak{o}(p)$ and $\mathfrak{o}(q)$ are compactly embedded, also $\mathfrak{k}$ is compactly embedded (the S is only a finite quotient).

(2) Show that for $(p, q) = (n, 1)$, the OSL $(\mathfrak{so}(n+1), \theta_{n,1})$ is associated to the symmetric space $\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$.

Remark: An OSL $(\mathfrak{g}, \theta)$ is *associated* to a symmetric space G/K given by a RSP (G, K) with involution σ, if $\mathfrak{g} = \operatorname{Lie}(G)$ and $\theta = \mathrm{D}_e\,\sigma$.

Solution:
Taking the derivative of ${}^tXX = \operatorname{Id}_{n+1}$ shows that $\operatorname{Lie}(\mathrm{SO}(n+1)) = \mathfrak{so}(n+1)$.

We choose $\mathrm{SO}(n) \cong \mathrm{SO}(n) \times \{1\} < \mathrm{SO}(n+1)$, then the Riemannian symmetric pair $(G, K) = (\mathrm{SO(n+1)}, \mathrm{SO}(n))$ has the involution

$$\begin{aligned} \sigma \colon G &\to G \\ g &\mapsto I_{n,1} g I_{n,1} = \left(\begin{array}{c|c} -\mathrm{Id}_n & 0 \\ \hline 0 & 1 \end{array}\right) g \left(\begin{array}{c|c} -\mathrm{Id}_n & 0 \\ \hline 0 & 1 \end{array}\right) \end{aligned}$$

and the derivative is $\mathrm{D}_e\, \sigma = \theta_{n,1}$, as can be seen by the following computation. Let $\gamma \colon \mathbb{R} \to G$ be a path with $\gamma(0) = e$ and $\dot\gamma(0) = X \in \mathfrak{so}(n+1)$.

$$\left.\frac{d}{dt}\right|_{t=0} \sigma(\gamma(t)) = I_{n,1} \left(\left.\frac{d}{dt}\right|_{t=0} \gamma(t) \right) I_{n,1} = \theta_{n,1}(X)$$

(3) Show that the complex dual of $(\mathfrak{so}(p{+}q), \theta_{p,q})$ is isomorphic to $(\mathfrak{so}(p,q), \theta_{p,q})$, where

$$\mathfrak{so}(p,q) := \left\{ X \in \mathfrak{gl}(n, \mathbb{R}) \colon I_{p,q}\, {}^t X + X I_{p,q} = 0 \right\}.$$

Remark: Two OSLs $(\mathfrak{g}_1, \theta_1)$ and $(\mathfrak{g}_2, \theta_2)$ are *isomorphic* if there is a Lie algebra isomorphism $\varphi \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ with $\varphi \circ \theta_1 = \theta_2 \circ \varphi$.

Solution: The complexification

$$\mathfrak{g} = \mathfrak{so}(p+q)^{\mathbb{C}} = \mathfrak{so}(p+q) + i \cdot \mathfrak{so}(p+q)$$

can be decomposed as

$$\mathfrak{g} = \mathfrak{k} + i\, \mathfrak{k} + \mathfrak{p} + i\, \mathfrak{p},$$

where $\mathfrak{k}$, resp. $\mathfrak{p}$ are the $+1$, resp -1 eigenspaces of $\theta_{p,q}$. Calculations show that

$$\begin{aligned} \mathfrak{k} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{gl}(n, \mathbb{R}) \colon A \in \mathfrak{so}(p), D \in \mathfrak{so}(q) \right\} = \mathfrak{s}(\mathfrak{o}(p) \times \mathfrak{o}(q)) \\ \mathfrak{p} &= \left\{ \begin{pmatrix} 0 & B \\ -\,{}^t B & 0 \end{pmatrix} \in \mathfrak{gl}(n, \mathbb{R}) \right\}, \end{aligned}$$

and we get the dual

$$\mathfrak{g}^* = \mathfrak{k} + i\, \mathfrak{p} = \left\{ \begin{pmatrix} A & iB \\ -i\,{}^t B & D \end{pmatrix} \in \mathfrak{gl}(n, \mathbb{C}) \colon A \in \mathfrak{so}(p), B \in \mathbb{R}^{p \times q}\, D \in \mathfrak{so}(q) \right\}$$

and claim there is a Lie algebra-isomorphism

$$\begin{aligned} \varphi \colon \qquad & \mathfrak{g}^* \to \mathfrak{so}(p,q) = \left\{ X \in \mathfrak{so}(p+q) \colon I_{p,q}\, {}^t X + X I_{p,q} = 0 \right\} \\ & \begin{pmatrix} A & iB \\ -i\,{}^t B & D \end{pmatrix} \mapsto \begin{pmatrix} A & B \\ {}^t B & D \end{pmatrix} \end{aligned}$$

This is an $\mathbb{R}$-linear bijective map, we need to check that the brackets are consistent:

$$\begin{aligned}
&\varphi\left(\left[\begin{pmatrix} A & iB \\ -i\,{}^tB & D \end{pmatrix}, \begin{pmatrix} \tilde{A} & i\tilde{B} \\ -i\,{}^t\tilde{B} & \tilde{D} \end{pmatrix}\right]\right) \\
&\quad = \varphi\left(\begin{pmatrix} A\tilde{A} + B\,{}^t\tilde{B} & iA\tilde{B} + iB\tilde{D} \\ -i\,{}^tB\tilde{A} - iD\,{}^t\tilde{B} & {}^tB\tilde{B} + D\tilde{D} \end{pmatrix} - \begin{pmatrix} \tilde{A}A + \tilde{B}\,{}^tB & i\tilde{A}B + i\tilde{B}D \\ -i\,{}^t\tilde{B}A - i\tilde{D}\,{}^tB & {}^t\tilde{B}B + \tilde{D}D \end{pmatrix}\right) \\
&\quad = \begin{pmatrix} A\tilde{A} + B\,{}^t\tilde{B} & A\tilde{B} + B\tilde{D} \\ {}^tB\tilde{A} + D\,{}^t\tilde{B} & {}^tB\tilde{B} + D\tilde{D} \end{pmatrix} - \begin{pmatrix} \tilde{A}A + \tilde{B}\,{}^tB & \tilde{A}B + \tilde{B}D \\ {}^t\tilde{B}A + \tilde{D}\,{}^tB & {}^t\tilde{B}B + \tilde{D}D \end{pmatrix} \\
&\quad = \left[\begin{pmatrix} A & B \\ {}^tB & D \end{pmatrix}, \begin{pmatrix} \tilde{A} & \tilde{B} \\ {}^t\tilde{B} & \tilde{D} \end{pmatrix}\right] \\
&\quad = \left[\varphi\begin{pmatrix} A & iB \\ -i\,{}^tB & D \end{pmatrix}, \varphi\begin{pmatrix} \tilde{A} & i\tilde{B} \\ -i\,{}^t\tilde{B} & \tilde{D} \end{pmatrix}\right].
\end{aligned}$$

The involution on $\mathfrak{g}^*$ is given by conjugation, which on $\mathfrak{k}$ is the identity and on $i\,\mathfrak{p}$ sends $X \mapsto -X$. When we translate this action via φ we get

$$\begin{pmatrix} A & B \\ {}^tB & D \end{pmatrix} = \varphi\begin{pmatrix} A & iB \\ -i\,{}^tB & D \end{pmatrix} \mapsto \varphi\begin{pmatrix} A & -iB \\ i\,{}^tB & D \end{pmatrix} = \begin{pmatrix} A & -B \\ {}^tB & D \end{pmatrix} = \theta_{p,q}\begin{pmatrix} A & B \\ {}^tB & D \end{pmatrix},$$

so the dual OSL of $(\mathfrak{so}(p+q), \theta_{p,q})$ is isomorphic to $(\mathfrak{so}(p,q), \theta_{p,q})$.

(4) Show that for $(p,q) = (n,1)$, the OSL $(\mathfrak{so}(n,1), \theta_{n,1})$ is associated to the symmetric space $\mathbb{H}^n = \mathrm{SO}(n,1)/\,\mathrm{SO}(n)$, where

$$\mathrm{SO}(p,q) := \left\{ g \in \mathrm{GL}(n,\mathbb{R}) \colon I_{p,q}\,{}^tgI_{p,q} = g^{-1} \right\}.$$

Solution: The Lie algebra of $\mathrm{SO}(p,q)$ consists of the elements $X \in \mathfrak{gl}(p+q)$ that satisfy $\exp(tX) \in \mathrm{SO}(p,q)$ for all $t \in \mathbb{R}$, i.e. $I_{p,q}\,{}^t\!\exp(tX)I_{p,q} = \exp(-tX)$. This is equivalent to $I_{p,q}XI_{p,q} = -X$ (by taking the derivative on both sides). It is also equivalent to $I_{p,q}X = -XI_{p,q}$ and in turn $I_{p,q}X + XI_{p,q} = 0$. So $\mathfrak{so}(p,q)$ is the Lie algebra of $\mathrm{SO}(p,q)$.
We have $\mathrm{SO}(n) \cong \mathrm{SO}(n) \times \{1\} < \mathrm{SO}(n,1)$ and the involution is given by

$$\begin{aligned}
\sigma \colon \mathrm{SO}(n,1) &\to \mathrm{SO}(n,1) \\
g &\mapsto \left(I_{n,1}\,{}^tgI_{n,1}\right)^{-1} = I_{n,1}gI_{n,1}
\end{aligned}$$

since $\mathrm{SO}(p,q) < \mathrm{O}(p+q)$ and the derivative is $\mathrm{D}_e\,\sigma = \theta_{n,1}$ as in (2).

Exercise 2: CAT(0)-spaces

Let M be a complete CAT(0)-space. Let C be a (non-empty) closed, convex subset of M. Prove the following:
Prop V.2(2): For any point $x \in M$ there exists a unique point $\pi_c(x) \in C$ with the property that $d(x, \pi_C(x)) \leq d(x,y)$ for any $y \in C$.

Solution: Let x be the point that we want to project on C. We consider a sequence of points x_i with $d(x,x_i) \to d(x,C)$ as $i \to \infty$. We want to show

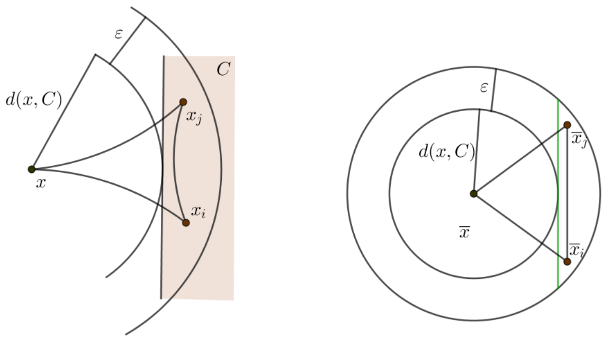

Figure 1: On the left, we have the triangle x, x_i, x_j in the CAT(0) space. On the right we have the comparison triangle. We note that $d(\overline{x}_i, \overline{x}_j)$ can at most be the length of the green line, which can be calculated explicitly.

that x_i is a Cauchy-sequence. So let $\varepsilon > 0$. There exists an $N > 0$ such that $d(x, x_i) \leq d(x, C) + \varepsilon$ for all $i \geq N$. Consider two points x_i, x_j with $i, j \geq N$. Now consider the comparison triangle $\overline{\Delta}(\overline{x}\overline{x}_i\overline{x}_j)$ of the triangle $\Delta(xx_ix_j)$. This is visualized in figure 1. Since C is convex, all points on the geodesic between x_i and x_j lie in C, so in the comparison triangle they also need to lie in the annulus between $d(x, C)$ and $d(x, C) + \varepsilon$. A calculation in $\mathbb{R}^2$ shows that such a straight line segment (green line in the figure) can have at most size $2\sqrt{d(x, C) + \varepsilon^2 - d(x, C)^2}$, therefore also $d(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ can have at most this distance and as ε goes to 0, so does the distance $d(x_i, x_j)$.

We have shown that $\{x_i\}$ is a Cauchy sequence, so since the space is complete, there exists a limit point, which we call $\pi(x)$. Since C is closed and all $x_i \in C$, also $\pi(x)$ is in C. By construction $d(x, \pi(x)) = d(x, C)$. We have to show uniqueness:

Let y and y' be two points with minimal distance $d(x, y) = d(x, y') = d(x, C)$. Consider the comparison triangle $\overline{\Delta}(\overline{x}, \overline{y}, \overline{y}')$. Since $d(\overline{x}, \overline{y}) = d(x, y) = d(x, y') = d(\overline{x}, \overline{y}')$, $\overline{\Delta}$ is isosceles. Now the midpoint z of y and y' on the unique geodesic between y and y' is in C, since C convex. We also have $\overline{z}$ on the line-segment from $\overline{y}$ to $\overline{y}'$. If $y \neq y'$, then $\overline{z} \neq \overline{y}$ is closer to $\overline{x}$ than $\overline{y}$, i.e. $d(\overline{z}, \overline{x}) < d(\overline{y}, \overline{x})$, thus by the CAT(0)-property also $d(x, z) \leq d(\overline{x}, \overline{z}) < d(\overline{y}, \overline{x}) = d(x, y)$, but that is impossible since $z \in C$ and $d(x, z)$ is the minimal distance from x to all points in C. We conclude that $y = y'$ and thus the projection π_C is well-defined.

Exercise 3: Dimension and Rank

(1) Calculate the dimension and rank of the symmetric space $\mathrm{SO}(p, q)/\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$.

Solution: From exercise 1, we know that

$$\mathrm{Lie}(\mathrm{SO}(p, q)) = \mathfrak{so}(p, q) = \left\{ \begin{pmatrix} A & B \\ {}^tB & D \end{pmatrix} \in \mathfrak{so}(p + q) \colon A \in \mathfrak{o}(p), D \in \mathfrak{o}(q) \right\}$$

We have the Cartan-decomposition:

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{so}(p+q) \colon A \in \mathfrak{o}(p), D \in \mathfrak{o}(q) \right\}$$
$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ {}^tB & 0 \end{pmatrix} \in \mathfrak{so}(p+q) \colon \right\}$$

The dimension of a manifold is the dimension of it's tangent-space, which in our case is isomorphic to $\mathfrak{p}$. Since we can choose $B \in \mathbb{R}^{p\times q}$ arbitrarily in $\mathfrak{p}$, $\mathfrak{p}$ and thus the symmetric space is $p \cdot q$-dimensional.
The rank of a symmetric space is the dimension of a maximal flat, so we need to find abelian subspaces of $\mathfrak{p}$. Consider

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & X \\ {}^tX & 0 \end{pmatrix} \in \mathfrak{p} \colon X \in \mathbb{R}^{p\times q}, X_{ij} = 0 \ \forall i \neq j \right\},$$

then a calculation

$$\left[\begin{pmatrix} 0 & X \\ {}^tX & 0 \end{pmatrix}, \begin{pmatrix} 0 & Y \\ {}^tY & 0 \end{pmatrix} \right] = \begin{pmatrix} X\,{}^tY - Y\,{}^tX & 0 \\ 0 & {}^tXY - {}^tYX \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

shows that $\mathfrak{a}$ is abelian (Use the fact that X, Y in the definition of $\mathfrak{a}$ are like diagonal matrices). To show that $\mathfrak{a}$ is a maximal subspace we have to find a regular element. For this choose

$$H = \begin{pmatrix} 0 & X \\ {}^tX & 0 \end{pmatrix} \in \mathfrak{a}$$

with distinct non-zero X_{ii} for $i \leq \min\{p, q\}$. For example for $(p, q) = (2, 3)$, this could look as follows:

$$H = \left(\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right), \quad \text{for} \quad X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

We claim that H is a regular element. Since $H \in \mathfrak{a}$ and $\mathfrak{a}$ abelian, $\mathfrak{a}$ is a subset of the centralizer $\mathfrak{z}(H)$ of H. Now let

$$\begin{pmatrix} 0 & Y \\ {}^tY & 0 \end{pmatrix} \in \mathfrak{z}(H), \quad \text{then}$$
$$0 = \left[H, \begin{pmatrix} 0 & Y \\ {}^tY & 0 \end{pmatrix} \right] = \begin{pmatrix} X\,{}^tY - Y\,{}^tX & 0 \\ 0 & {}^tXY - {}^tYX \end{pmatrix},$$

so $X\,{}^tY = Y\,{}^tX$ and ${}^tXY = {}^tYX$, which can be rewritten as $X\,{}^tY\,{}^tX^{-1} = Y$ and ${}^tXYX^{-1} = {}^tY$, so $Y = X\,{}^tY\,{}^tX^{-1} = X\,{}^tXYX^{-1}\,{}^tX^{-1} = (X\,{}^tX)Y({}^tXX)^{-1}$, where $X\,{}^tX$ and $({}^tXX)^{-1}$ are diagonal matrices. Then $Y_{ij} = X_{ii}^2 Y_{ij} X_{jj}^{-2}$, which has to be zero for $i \neq j$, because X_{ii} are distinct and non-zero for $i \leq \min\{p, q\}$. This shows that $\mathfrak{a} = \mathfrak{z}(H) \cap \mathfrak{p}$, so H is a regular element and $\mathfrak{a}$ is maximal abelian. The dimension of $\mathfrak{a}$ is $\min\{p, q\}$, so the rank of the symmetric space $\mathrm{SO}(p, q)/\,\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$ is $\min\{p, q\}$.

Consider the Lie group of symplectic matrices

$$\mathrm{Sp}(2n,\mathbb{R}) = \left\{T \in \mathrm{GL}(2n,\mathbb{R})\colon\ {}^tTI_nT = I_n\right\}, \quad \text{where} \quad I_n = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}.$$

with involution $\sigma\colon \mathrm{Sp}(2n,\mathbb{R}) \to \mathrm{Sp}(2n,\mathbb{R})$ given by $g \mapsto {}^tg^{-1}$.

(2) Show that the fixed point set of σ is isomorphic to $\mathrm{U}(n)$ and show that $(\mathrm{Sp}(2n), \mathrm{U}(n))$ is a Riemannian symmetric pair.

$$\mathrm{U}(n) = \left\{Z = X + iY \in \mathrm{GL}(n,\mathbb{C})\colon\ {}^t\overline{Z}Z = \mathrm{Id}\right\}$$

Hint: Consider $g^{-1} = {}^tg$. *Assume without proof that* $\mathrm{Sp}(2n,\mathbb{R})$ *is connected.*

Solution: Let

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R}), \quad \text{i.e.}$$
$${}^tAC = {}^tCA, \quad {}^tBD = {}^tDB, \quad {}^tAD - {}^tCB = \mathrm{Id}_n$$

We note that

$$\begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} {}^tDA - {}^tBC & {}^tDB - {}^tBD \\ {}^tAC - {}^tCA & {}^tAD - {}^tCB \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & \mathrm{Id}_n \end{pmatrix},$$

so the condition $\sigma(g) = {}^tg^{-1} = g$ of being a fixed point of σ, turns into

$${}^tg^{-1} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

from which we conclude that elements $g \in \mathrm{Sp}(2n,\mathbb{R})$ that are fixed by σ are exactly the matrices of the form

$$g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

which satisfy ${}^tBA = {}^tAB$ and ${}^tAA + {}^tBB = \mathrm{Id}_n$. We set up an isomorphism

$$\varphi\colon\ \mathrm{Sp}(2n,\mathbb{R})^\sigma \to U(n)$$
$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + iB,$$

indeed this is well defined since ${}^t\overline{(A+iB)}(A+iB) = ({}^tA - i\,{}^tB)(A+iB) = {}^tAA + {}^tBB + i({}^tAB - {}^tBA) = \mathrm{Id}_n$. The map is also a group-homomorphism because

$$\begin{aligned}
\varphi\left(\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \cdot \begin{pmatrix} A' & B' \\ -B' & A' \end{pmatrix}\right) &= \varphi\begin{pmatrix} AA' - BB' & AB' + BA' \\ -AB' - BA' & AA' - BB' \end{pmatrix} \\
&= AA' - BB' + i(AB' + BA') \\
&= (A + iB)(A' + iB') \\
&= \varphi\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \cdot \varphi\begin{pmatrix} A' & B' \\ -B' & A' \end{pmatrix}
\end{aligned}$$

and φ^{-1} is well defined, so φ is an isomorphism.
Now $\mathrm{Sp}(2n,\mathbb{R})$ is a connected group with a closed (since defined by equalities) subgroup, which is isomorphic to $\mathrm{U}(n)$. Since $\mathrm{U}(n)$ is compact, so is $\mathrm{Ad}(\mathrm{U}(n))$. Since $(\mathrm{Sp}(2n,\mathbb{R})^\sigma)^\circ \subset \mathrm{U}(n) \cong \mathrm{Sp}(2n,\mathbb{R})^\sigma$, $(\mathrm{Sp}(2n,\mathbb{R}),\mathrm{U}(n))$ is a Riemannian symmetric pair.

(3) Calculate the dimension and rank of the symmetric space $\mathrm{Sp}(2n,\mathbb{R})/\,\mathrm{U}(n)$.

Solution: We consider the Lie algebra

$$\mathfrak{sp}(2n,\mathbb{R}) = \left\{X \in \mathfrak{gl}(2n,\mathbb{R})\colon I_nX + {}^tXI_n = 0\right\},$$

which we get by taking the derivative of the condition ${}^tgI_ng = I_n$. These are exactly the matrices

$$\begin{aligned} X &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{with} \\ 0 = I_nX + {}^tXI_n &= \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD \end{pmatrix}\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \\ &= \begin{pmatrix} C - {}^tC & D + {}^tA \\ -(A + {}^tD) & {}^tB - B \end{pmatrix}, \end{aligned}$$

so B and C symmetric and $D = -\,{}^tA$. For B and C there are $n(n+1)/2$ degrees of freedom, A has all n^2 degrees of freedom, but D is totally determined by A. We get that $\mathfrak{sp}(2n,\mathbb{R})$ and therefore $\mathrm{Sp}(2n,\mathbb{R})$ have dimension $2n^2+n$.
The Lie algebra $\mathfrak{u}(n)$ of $\mathrm{U}(n)$ is given by

$$\mathfrak{u}(n) = \left\{X \in \mathfrak{gl}(n,\mathbb{C})\colon\ {}^t\overline{X} + X = 0\right\},$$

which means that the diagonal entries $X_{ii} = a+bi$ satisfy $0 = \overline{X}_{ii} + X_{ii} = a - bi + a + bi = 2a$, so the real part a of X_{ii} is 0. The imaginary part of the diagonal entries is arbitrary. The strict upper triangular part determines the strict lower triangular part, which implies that there are $n(n-1)$ degrees of freedom in the strict upper triangular part. This implies that the real dimension of $\mathfrak{u}(n)$ and therefore of $\mathrm{U}(n)$ is $n + n(n-1) = n^2$.
The dimension of the symmetric space $\mathrm{Sp}(2n,\mathbb{R})/\,\mathrm{U}(n)$ is thus $2n^2+n-n^2 = n^2+n$.
For the cartan decomposition $\mathfrak{sp}(2n,\mathbb{R}) = \mathfrak{u}(n) \oplus \mathfrak{p}$, we get

$$\begin{aligned} \mathfrak{p} &= \left\{X \in \mathfrak{sp}(2n,\mathbb{R})\colon\ -\,{}^tX = X\right\} \\ &= \left\{\begin{pmatrix} A & B \\ {}^tB & D \end{pmatrix} \in \mathfrak{sp}(2n,\mathbb{R})\colon A = {}^tA,\ D = {}^tD\right\} \end{aligned}$$

For the rank we consider the abelian subspace

$$\mathfrak{a} = \left\{\begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \in \mathfrak{sp}(2n,\mathbb{R})\colon X \text{ diagonal}\right\}$$

and choose a special element $H \in \mathfrak{a}$ whose X has distinct positive, non-zero entries. Since $\mathfrak{a}$ is abelian and $H \in \mathfrak{a}$, we have $\mathfrak{a} \subset \mathfrak{z}(H)\cap\mathfrak{p}$. Now let $Y \in \mathfrak{z}(H)\cap\mathfrak{p}$, i.e. $[H,Y]=0$. This implies

$$0 = [H,Y] = \left[\begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}, \begin{pmatrix} A & B \\ {}^tB & D \end{pmatrix}\right] = \begin{pmatrix} XA - AX & XB + BX \\ -\,{}^tBX - X\,{}^tB & CX - XC \end{pmatrix}$$

Considering the entries of $XB + BX = 0$, we get $0 = (XB + BX)_{ij} = X_{ii}B_{ij} + X_{jj}B_{ij} = (X_{ii} + X_{jj})B_{ij}$ and since the entries of X are positive, $B_{ij} = 0$ and thus $B = 0$. Similarly $0 = (XA - AX)_{ij} = (X_{ii} - X_{jj})A_{ij}$ and since the entries of X are distinct, $A_{ij} = 0$ for all $i \neq j$. Thus A is a diagonal matrix. Now we use the condition $D + {}^tA = 0$, which holds since $Y \in \mathfrak{sp}(2n, \mathbb{R})$. Since A is diagonal, we get $D = -A$ and thus $Y \in \mathfrak{a}$. We have shown that $\mathfrak{z}(H) \cap \mathfrak{p} = \mathfrak{a}$ is an abelian, and thus maximal abelian (H is a regular element). The dimension of $\mathfrak{a}$ is n, so the rank of the symmetric space $\mathrm{Sp}(2n, \mathbb{R}) / \mathrm{U}(n)$ is also n.