Definition 2.8. Let $M$ be a connected Riemannian manifold. Then $M$ is locally symmetric if for all $p \in M$ there is a normal neighbourhood $U$ of $p \in M$ and an isometry $s_p : U \to U$ such that $s_p^2 = \text{id}$ and such that $p$ is the unique fixed point of $s_p$ in $U$. The space $M$ is globally symmetric if it is locally symmetric and each $s_p$ can be extended to an isometry of $M$.

Ex: $S^2$, $E^2$, $S^2$, $H^2$, $S^1 \times S^1$, $R \times S^1$, Klein bottle, $RP^2$.

2-dim RSS: $E^2$, $S^2$, $H^2$, $S^1 \times S^1$, $R \times S^1$, Klein bottle, $RP^2$.

$R^2 \xrightarrow{s_p} R^2$.

$f = s_p \circ \pi \quad \forall x \in R^2$.

If $A \in G$: $f(A(x)) = f(x)$.

Then $s_p$ is well defined.

$M = \frac{H^2}{\Gamma}$.

(Note: There may be more than one global fixed point of $s_p$.)

Euclidean type: compact type.

$\mathbb{R}_+$

One locally symmetric.

$M$ locally symmetric space, $\tilde{M} / \Gamma = M$.

Ex: $S^n$:

$S^2 = \mathbb{R}^2 / \mathbb{Z}^2$.

Def. 2.31. Let $G$ be connected Lie group and let $K \leq G$ be a closed subgroup of $G$. Then $(G, K)$ is a Riemannian symmetric pair if

(i) $\text{Ad}_G(K) \leq \text{GL}(g)$ is compact, and

(ii) there exists an involution $\sigma : G \to G$ with $(G^\sigma)^o \subset K \subset G^\sigma$.

Note: If $K$ is compact, then $\text{Ad}(K)$.

Ex: $S^n$:

$S^n$.

$S^2 = \mathbb{R}^2 / \mathbb{Z}^2$.

$S^2 \to \mathbb{R}P^2$.

RSP
\[ S^n \Rightarrow \mathbb{R}^n \]

\[ S_0 : S^n \to S^n \]

\[ p \mapsto (I_{n-1})_p \]

\[ C_n := I_{n-1}(S^n)^0 = O(n+1)^0 = SO(n+1) \]

\[ K := \text{Stab}_C(o) = \{ g \in SO(n+1) : ge_{uu} = e_{uu} \} = \{ (A \ 0) : A \in SO(n) \} \]

\[ g = \begin{pmatrix} A & \rho \\ 0 & 1 \end{pmatrix}, \quad ge_{uu} = e_{uu} \iff \begin{pmatrix} B \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \iff d = 1, \quad B = 0. \]

\[ \begin{pmatrix} A^T & C^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ C & 1 \end{pmatrix} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 1 \end{pmatrix} \]

\[ G = \{ (A \ 0) : A \in SO(n) \} \]

\[ G^{\sigma} = \{ (A \ 0) : A \in SO(n) \} \]

\[ G^o = \{ (A \ 0) : A \in SO(n) \} \]

\[ G^{\sigma} = \{ (A \ 0) : A \in SO(n) \} \]

\[ (G^{\sigma})^0 = \{ (A \ 0) : A \in SO(n) \} = K \subseteq G^{\sigma}. \]

Ex: \( M = \mathbb{R}P^2 = S^2 / \sim \)

\[ 0 = e_3, \quad s_0 : \mathbb{R}P^2 \to \mathbb{R}P^2 \]

\[ G = I_{S}(M)^0 = \left( O(3) \right)^0_{\sim / \Id} = O(3)^0_{\sim / \Id} \]

\[ K = \text{Stab}_C(o) = \{ g \in O(3)^0_{\sim / \Id} : g_0 = o \} = \{ [(A \ 0)_{\sim / \Id}] : A \in O(3) \} \]

\[ G^{\sigma} = \{ (A \ 0)_{\sim / \Id} : (S_0 \cdot g : S_0 = g) = K \]

\[ \Rightarrow (G^{\sigma})^0 \subseteq K = G^{\sigma}. \]

Ex: \( (G^{\sigma})^0 \neq K \neq G^{\sigma} \) for \( M = S^2 \times \mathbb{R}P^2 \)

\[ G^{\sigma} / K = M \] and Riemannian metric is determined by a
\[ W/K = M \] and Riemannian metric is determined by a 
\[ \text{Ad}(K) \]-invariant scalar product on \[ \mathfrak{p} = T_0 M \].

**Definition 2.53.** Let \( \mathfrak{g} \) be a Lie algebra. A subalgebra \( \mathfrak{t} \leq \mathfrak{g} \) is compactly embedded if \( \text{ad}_{\mathfrak{g}}(\mathfrak{t}) \) is the Lie algebra of a compact subgroup of \( \text{GL}(\mathfrak{g}) \).

**Definition 2.54.** An orthogonal symmetric Lie algebra is a pair \((\mathfrak{g}, \Theta)\) consisting of a real Lie algebra \( \mathfrak{g} \) and an involutive automorphism \( \Theta \neq \text{Id} \) of \( \mathfrak{g} \) such that \( u = E_1(\Theta) \) is compactly embedded in \( \mathfrak{g} \). An orthogonal symmetric Lie algebra is effective if \( Z(\mathfrak{g}) \cap u = 0 \).

**Del:** \((\mathfrak{g}, \Theta)\) is reduced if there is no non-zero ideal of \( \mathfrak{g} \) in \( \mathfrak{t} \).

**RSP** \(\rightarrow\) **OSL**

\[ \mathfrak{g} = \text{Lie}(\mathfrak{g}), \Theta = D^\sigma. \]

**OSL** \(\rightarrow\) **RSP**

\((\mathfrak{g}, \Theta)\) **OSL** (effective), semi-simple. Can choose \( \mathfrak{g} = \text{Aut}(\mathfrak{g})^0, \sigma: \mathfrak{g} \rightarrow \mathfrak{g}, (\mathfrak{g}^0)^0 < K < \mathfrak{g}^0. \)

If \( \mathfrak{g} \) is semi-simple: Then \( \text{Aut}(\mathfrak{g})^0 \cong \text{Ad}(\mathfrak{g}) \Rightarrow \mathfrak{g} \cong \text{Lie}(\mathfrak{g}) \)
effective: \( \mathfrak{z}(\mathfrak{g}) = 0 \)

\[ \Rightarrow \mathfrak{g} = \text{Aut}(\mathfrak{g})^0. \]

\[ \text{Lie}(\mathfrak{g}) = \mathfrak{g}, \text{ then } \text{Aut}(\mathfrak{g})^0 \cong \text{Ad}(\mathfrak{g}) \Rightarrow \text{Lie}(\mathfrak{g}) = \text{Lie}(\mathfrak{g}). \]

"Injectivity" for **RSS** \(\rightarrow\) **OSL**

Let \( M_1, M_2 \) be two **RSS** of non-compact type.

\[ \Rightarrow \text{OSL's } (\mathfrak{g}_1, \Theta_1), (\mathfrak{g}_2, \Theta_2). \]

If \( (\mathfrak{g}_1, \Theta_1) \cong (\mathfrak{g}_2, \Theta_2) \) isomorphic, \( M_1 \) and \( M_2 \) have the metric coming from the \( B_\mathfrak{g} \).

\[ \Rightarrow M_1 \text{ is Riemannian isometric to } M_2. \]

**Definition 2.57.** Let \((\mathfrak{g}, \Theta)\) be an effective orthogonal symmetric Lie algebra with Cartan decomposition \( \mathfrak{g} = \mathfrak{u} \oplus \mathfrak{c} \) and Killing form \( B_\mathfrak{g} \).

(i) If \( B_\mathfrak{g} \ll 0 \) then \((\mathfrak{g}, \Theta)\) is of compact type.
Definition 2.51. Let \((g, \Theta)\) be an effective orthogonal symmetric Lie algebra with Cartan decomposition \(g = u \oplus \mathfrak{c}\) and Killing form \(B_g\).

(i) If \(B_g \ll 0\) then \((g, \Theta)\) is of compact type.
(ii) If \(B_g\) is non-degenerate and \(B_g|_{\mathfrak{c} \times \mathfrak{c}} \gg 0\) then \((g, \Theta)\) is of non-compact type.
(iii) If \(\mathfrak{c}\) is an abelian ideal in \(g\) then \((g, \Theta)\) is of Euclidean type.

\[ g = \mathfrak{k} \oplus \mathfrak{p} = u \oplus \mathfrak{c} = E_n(\Theta) \oplus E_n(\Theta). \]

If effective \(\Rightarrow B_g|_{\mathfrak{k} \times \mathfrak{k}} \ll 0\).

Theorem 2.60. Let \((g, \Theta)\) be an effective orthogonal symmetric Lie algebra. Then \(g = g_0 \oplus g_+ \oplus g_-\) is a direct sum of \(\Theta\)-stable ideals such that:

(i) The \(g_\mu\) (\(\mu \in \{0, +, -\}\)) are mutually orthogonal with respect to \(B_g\).
(ii) The pairs \((g_0, \Theta|_{g_0})\), \((g_+, \Theta|_{g_+})\), \((g_-, \Theta|_{g_-})\) are orthogonal symmetric Lie algebras of Euclidean, non-compact and compact type respectively.

Choose a scalar product \(\langle , \rangle\) on \(\mathfrak{p}\).

\[ \langle B_g(x, y) \rangle = \langle AX, Y \rangle, \quad A \text{ is symmetric.} \]

\(\Rightarrow\) diagonalize \(A\), look at sign of eigenvalues of \(A\),
\(\Rightarrow\) the corresponding eigenspaces = \(p_0, p_-\), \(p_+\).

Theorem 2.73. Let \((g, \Theta)\) be an orthogonal symmetric Lie algebra with Cartan decomposition \(g = \mathfrak{c} \oplus \mathfrak{p}\) such that \(g\) is semisimple and \(\mathfrak{c}\) does not contain an ideal of \(g\). Then there are ideals \((g_\mu|_{\mathfrak{c}})\) of \(g\) such that

(i) \(g = \bigoplus \mathfrak{c}_\mu\),
(ii) the \((g_\mu|_{\mathfrak{c}})\) are pairwise orthogonal with respect to \(B_g\) and \(\Theta\)-invariant, and
(iii) \((g_\mu, \Theta|_{g_\mu})\) is an irreducible orthogonal symmetric Lie algebra.

Moreover, this decomposition is unique.

Globalize decomposition.

M \(\Rightarrow\) OSL \((g, \Theta)\), reduced. \(\mathbb{R}^n\)

If \(M\) is simply connected \(\Rightarrow M = M_0 \times M_+ \times M_- = \mathbb{R}^n \times \Sigma M_+\).

\[ \text{but } b \text{ be a bilinear form, } b(x, y) = x^T (I_n A) y \]

on \(\mathbb{R}^{n+1}\)

\[ H^{n+1} = \{ x \in \mathbb{R}^{n+1} : b(x, x) = -1, \ x_{n+1} > 0 \} \]

\[ \mathcal{I}_{g}(H^{n+1}) = \{ g \in M_{n+1} : b(gx, gy) = b(x, y) \forall x, y \in H^{n+1} \} \]
\[ \begin{align*}
I_s(H^m) &= \{ g \in M_{n+1} : b(g x, g y) = b(x, y) \ \forall x, y \in H^m \} \\
0 &= e_{n+1} \\
S_0 : H^m &\rightarrow H^m \\
p &\mapsto (-I_{n+1}) p \\
C &= I_s(H^m)^0 = O(n,1)^0 = \alpha_{n,1}.
\end{align*} \]

\[ g_0 = \text{Lie}(\gamma) = \alpha(\nu,1) = \{ X \in M_{n+1} : X^T (-I_{n+1}) + (-I_{n+1}) X = 0 \} \]

\[ \Theta = D_{\sigma}, \quad \sigma(q) = s_0 g s_0 = (-I_{n+1}) g (-I_{n+1}) \]

\[ \Theta(x) = (-I_{n+1}) X (-I_{n+1}) \]

\[ \Theta(A, B) = (A - B) = (A + B) \]

\[ \begin{align*}
\Theta(C, D) &= (-C, D) \\
\Theta(C, D) &= (C, D)
\end{align*} \]

\[ \begin{align*}
E_{\gamma}(\theta) &= \{ X \in \alpha(n,1) : \Theta(X) = X \} = \{ (A, 0) : A^T + A = 0 \} \\
p = E_{\gamma}(\theta) &= \{ (0, B) \in \alpha(n,1) : B \in C_s \} = \{ (0, B) ; B \in R^{n+1} \} \\
\Theta &= (0, B)^T (-I_{n+1}) + (-I_{n+1}) (0, B) = (-B^T, C_s) + (0, -B^T, 0) \\
\end{align*} \]

\[ \bar{g}_0 = k \oplus p, \quad \text{Complexification} \quad \bar{g} = g_0 + i \cdot g_0 = k \oplus p + i \cdot k \oplus i \cdot p \]

\[ g = \alpha(n,1) + i \cdot \alpha(n,1). \]

\[ g^* := k \oplus i \cdot p = \{ (A, iB) : A^T + A = 0 \} \]

Thus, \( (g_0, \Theta)^* = (g^*, \text{complex cay}) \) is again an OSL.

to be continued...