II.5 Exponential maps and geodesics.

Let M be a riemannian symmetric space, $o \in M$, $G = Is(M)^{\circ}$ and $K = Stab_G(o)$. Let $\mathfrak{g}$ be the Lie algebra of G and $\exp : \mathfrak{g} \longrightarrow G$ the (Lie group) exponential. On the other hand we also have the riemannian exponential

$$Exp_o : T_o M \longrightarrow M.$$

The following result establishes a direct link between those two exponentials. Recall $\pi : G \longrightarrow M$, $g \longmapsto g.o$ induces a vector space isomorphism

$$D_e\pi : \mathfrak{p} \longrightarrow T_o M.$$

Theorem II.39 Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ (Def. II.37). Then the diagramm

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{D_e\pi} & T_oM \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \mathrm{Exp}_o} \\ G & \xrightarrow[\pi]{} & M \end{array}$$

commutes.

In particular for every $Y \in \mathfrak{p}$, $t \longmapsto \exp(tY)_* o$ is the geodesic passing through o with velocity $D_e\pi(Y) \in T_oM$.

~~Exercise~~.

Proof:

Let $X \in \mathfrak{p}$, then $\gamma(t) := \mathrm{Exp}_o\left(t\, D_e\pi(X)\right)$ is a geodesic through o with $\dot\gamma(0) = D_e\pi(X) \in T_oM$. Let

$$\tau_t := s_{\gamma(\frac{t}{2})}\, s_o$$

be the 1-parameter group of transvections associated to the geodesic γ (see Prop. II.22).

By Lie group theory there is $Y \in \mathfrak{g}$ such that

$$\tau_t = \exp(tY) \quad \forall t \in \mathbb{R}.$$

Recall that the involution on G is given by $\sigma(g) = s_o\, g\, s_o$ and $\Theta = D_e\sigma \in \mathrm{Aut}(\mathfrak{g})$ is the Cartan involution.

Claim $Y \in \mathfrak{p}$: we have $\forall t \in \mathbb{R}$

$$\exp(t\,\Theta(Y)) = \sigma(\exp tY) = s_o\, \tau_t\, s_o$$

$$= s_o\, s_{\gamma(\frac{t}{2})}\, s_o^2 = s_o\, s_{\gamma(\frac{t}{2})}$$

$$= s_o^{-1}\, s_{\gamma(\frac{t}{2})}^{-1} = \left(s_{\gamma(\frac{t}{2})}\, s_o\right)^{-1}$$

$$= \tau_t^{-1} = \tau_{-t} = \exp(-tY)$$

which implies $\Theta(Y) = -Y$ and hence

$Y \in \mathfrak{p}$.

From $\pi\, \tau_t = \tau_t . \gamma(0) = \gamma(t)$ we

deduce by taking the derivative at $t=0$:

$$D_e\pi\left(\frac{d}{dt}\Big|_{t=0} \tau_t\right) = \dot{\gamma}(0)$$

that is $D_e\pi\,(Y) = D_e\pi\,(X)$ which

implies $Y = X$. Then

$$\pi\, \tilde{\gamma}_t = \gamma(t)$$

is precisely :

$$\pi(\exp tX) = \mathrm{Exp}_o\, t\, D_e\pi(X).$$

$\boxed{\text{W}}$

As an application we are going to deduce a formula for the derivative of $\mathrm{Exp}_o : T_oM \longrightarrow M$ at an arbitrary point of T_oM ; this formula will be important in the study and characterization of totally geodesic sub-manifolds of M. First we recall the classical formula for the derivative of the exponential map of a

Lie group. A simple proof can be found in Sternberg "Lie algebras" which is available as pdf on the homepage (Section 1.5, p.15).

<u>Thm II.40</u> Let G be a Lie group $\exp : \mathfrak{g} \to G$ the exponential map and $X \in \mathfrak{g}$. Under identification of $T_X \mathfrak{g}$ with $\mathfrak{g}$ we have:

$$D_X \exp = D_e\left(L_{\exp X}\right)\left(\sum_{n=0}^{\infty} \frac{(\operatorname{ad} X)^n}{(n+1)!}\right).$$

Recall here that $\operatorname{ad} : \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ is the derivative at e of the adjoint representation $\operatorname{Ad} : G \to GL(\mathfrak{g})$.

Hence the relation between Ad and ad is :

$$Ad(\exp tX) = \exp(t\, ad(X)) \quad \forall t \in \mathbb{R},\ \forall X \in \mathfrak{g}$$

from which one deduces (see the Lie groups course) :

$$ad(X)(Y) = [X, Y], \quad X, Y \in \mathfrak{g}.$$

Coming back to the context of riemannian symmetric spaces and with the notations and concepts introduced before Thm II.39. Let's define

$$Exp : \mathfrak{p} \longrightarrow M$$

as the composition of $\mathfrak{p} \xrightarrow{D_e\pi} T_o M \xrightarrow{Exp_o} M$.

For $X \in \mathfrak{p}$ define $T_X = (\operatorname{ad} X)^2$.

Then it follows from Prop. I.38 that T_X preserves the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, in particular ~~sends~~ $\mathfrak{p}$ into $\mathfrak{p}$.

With these notations we have then the following formula for the derivative of $\operatorname{Exp} : \mathfrak{p} \longrightarrow TM$ at X:

<u>Corollary II.41</u> For $X \in \mathfrak{p}$ we have

$$D_X \operatorname{Exp} = \left(DL_{\exp X} \circ D_e \pi \right) \left(\sum_{n=0}^{\infty} \frac{\left(T_X|_{\mathfrak{p}} \right)^n}{(2n+1)!} \right)$$

Observe that $DL_{\exp X} \circ D_e \pi : \mathfrak{p} \longrightarrow T_{(\exp X).o} M$ is a vector space isomorphism ~~and~~

and by Thm II. 39:

$$(\exp X).o = \mathrm{Exp}_o(D_e\pi(X)) = \mathrm{Exp}(X).$$

<u>Proof</u>: We have by Thm II. 39: for $X \in \mathfrak{p}$.

$$D_X \mathrm{Exp} = D_X(\mathrm{Exp}_o \circ D_e\pi|_{\mathfrak{p}})$$

$$= D_X(\pi \circ \exp|_{\mathfrak{p}})$$

Now let's compute the derivative of $\pi \circ \exp : \mathfrak{g} \longrightarrow M$ at an arbitrary vector $X \in \mathfrak{g}$. We have:

$$D_X(\pi \circ \exp) = (D_{\exp X}\pi)\, D_X \exp$$

$$= D_{\exp X}\pi \circ D_e L_{\exp X}\left(\sum_{n=0}^{\infty} \frac{(\mathrm{ad}\, X)^n}{(n+1)!}\right)$$

the last equality being an application of

Thm II.40 and $L_{\exp} : G \longrightarrow G$ is left multiplication by exp, by abuse of notation since we have also denoted by $L_g : M \longrightarrow M$ the left multiplication by g. But since $\pi \circ L_g = L_g \circ \pi$ we deduce that

$$D_{\exp X}\pi \circ D_e L_{\exp X} = D_o L_{\exp X} D_e \pi$$

and hence

$$D_X(\pi \circ \exp) = D_o L_{\exp X} D_e\pi \left(\sum_{n=0}^{\infty} \frac{(\mathrm{ad}\, X)^n}{(n+1)!} \right).$$

Now let $X \in \mathfrak{p}$: then $\mathrm{ad}\, X(\mathfrak{p}) \subset \mathfrak{z}$ (see II.38); now we have

$$(\mathrm{ad}\, X)^{2n+1} = (T_X)^n \,\mathrm{ad}\, X$$

and since T_X, and hence T_X^n, preserves the Cartan decomposition, we have:

$$(\mathrm{ad}\, X)^{2n+1}(\mathfrak{p}) = T_X^n\, \mathrm{ad}X(\mathfrak{p}) \subset T_X^n(\mathfrak{k}) \subset \mathfrak{k}.$$

But since $\mathrm{Ker}\, D_e\pi = \mathfrak{k}$ we conclude that

$$(D_e\pi)(\mathrm{ad}\, X)^{2n+1}\Big|_{\mathfrak{p}} = 0$$

which concludes the proof of the formula. $\boxed{\xi}$

II. 6. Totally geodesic submanifolds.

In this section we will characterize totally geodesic submanifolds of a symmetric space in Lie algebraic terms. This will use the formula for the derivative of the exponential map obtained in Corollary II. 41.

Let $N \subset M$ be a connected submanifold of a riemannian manifold (M, g).

Def. II. 42 N is geodesic at $p \in N$ if $\forall v \in T_pN$ the M-geodesic through (p, v) is contained in N.