The submanifold N is totally geodesic if it is geodesic at every $p \in N$.

Let N be a connected submanifold of a riemannian manifold (M, g) and let $g|_N$ be the restriction of g to N as well as d_M resp. d_N the corresponding riemannian distance on M resp. N.

Then clearly $d_M(p,q) \leq d_N(p,q) \ \forall p,q \in N$.

<u>Exercise II.43</u>. Assume $N \subset M$ is totally geodesic.

(1) The inclusion $(N, d_N) \hookrightarrow (M, d_M)$ is locally distance preserving.

(2) Every N-geodesic is an M-geodesic and every M-geodesic contain in N is an N-geodesic.

The following is a characterization of being totally geodesic that we will use below:

<u>Prop. II.44</u> $N \subset M$ is totally geodesic if and only if the parallell transport for the riemannian metric on M along curves contained in N preserves the tangent space distribution $\{T_p N : p \in N\}$.

Example II.43 A totally geodesic manifold is not necessarily embedded. For instance take the 3-dimensional flat torus $\mathbb{T}^3 := \mathbb{Z}^3 \backslash \mathbb{E}^3$, with projection $\pi : \mathbb{E}^3 \longrightarrow \mathbb{T}^3$. It follows from the definition that if $P \subset \mathbb{E}^3$ is any say 2-dimensional plane then $\pi(P) \subset \mathbb{T}^3$ is totally geodesic. One can choose P in such a way that $\pi(P)$ is dense in $\mathbb{T}^3$.

The Lie algebraic concept underlying total geodesicity is:

Def. II. 44 A vector subspace $\mathfrak{n}$ of a Lie Algebra $\mathfrak{g}$ is a Lie triple system if $[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] \subset \mathfrak{n}$.

Prominent example is the subspace $\mathfrak{p}$ in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Indeed by II.38:

$$[[\mathfrak{p},\mathfrak{p}],\mathfrak{p}] \subset [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}.$$

<u>Lemma II.45</u> If $\mathfrak{n} \subset \mathfrak{g}$ is a Lie triple system then

(1) $[\mathfrak{n},\mathfrak{n}]$ is a subalgebra of $\mathfrak{g}$

(2) $\mathfrak{n} + [\mathfrak{n},\mathfrak{n}]$ is a subalgebra of $\mathfrak{g}$.

<u>Proof.</u>

(1) Let X, Y, Z, W in $\mathfrak{n}$. We apply the Jacobi identity to X, Y and $[Z,W]$:

$$[[X,Y],[Z,W]] + [[[Z,W],X],Y] + [[Y,[Z,W]],X]$$

$$= 0$$

But $[[Z,W],Y] \in \mathfrak{n}$, $[Y,[Z,W]] \in \mathfrak{n}$

which implies (1).

(2) If X, Y are in $\mathfrak{n}$ then $[X,Y] \in [\mathfrak{n},\mathfrak{n}]$.

If $X \in \mathfrak{n}$ and $Y \in [\mathfrak{n},\mathfrak{n}]$ then by hypothesis $[X,Y] \in [\mathfrak{n},[\mathfrak{n},\mathfrak{n}]] \subset \mathfrak{n}$.

Finally $[\mathfrak{n},\mathfrak{n}]$ is a subalgebra.

$\boxed{\xi}$

<u>Theorem II.46.</u>

Let $M \ni o$ be a symmetric space

$G = Is(M)^\circ$, $K = Stab_G(o)$, $\pi : G \longrightarrow M$

the canonical projection and

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{p}$$

the Cartan decomposition. Then the following hold:

(1) If $\mathfrak{n} \subset \mathfrak{p}$ is a Lie triple system then $\mathrm{Exp}_o(D_e\pi(\mathfrak{n})) \subset M$ is a totally geodesic submanifold, containing $o \in M$.

(2) If $N \subset M$ is a totally geodesic submanifold containing o, then $\mathfrak{n} := (D_e\pi)^{-1}(T_oN) \subset \mathfrak{p}$ is a Lie triple system.

<u>Remark II.47</u> If $P \subset M$ is an arbitrary totally geodesic manifold then pick $p \in P$ and $g \in G$ with $L_g(p) = o$; then $L_g(P) \ni o$ is totally geodesic as well and (2) Thm II.46 applies.

<u>Proof.</u>

(1) Let $\mathfrak{g}' := \mathfrak{n} + [\mathfrak{n}, \mathfrak{m}]$; then (lemma II.45) $\mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}$. Let $G' < G$ be the corresponding Lie subgroup, and $K' := G' \cap K$. Then it follows from the definition of smooth structures on G'/K' and G/K that the inclusion $G'/K' \hookrightarrow G/K$ is an immersive map. As a result if $M' := G'_* o$ the map

$$G'/K' \longrightarrow M$$
$$gK' \longmapsto g_* o$$

is immersive with image M' and M' is thus a submanifold of M.

Therefore : $D_e\pi(\mathfrak{g}'\cap\mathfrak{p}) = T_o M'$; but from $\mathfrak{g}' = \mathfrak{n} + [\mathfrak{n},\mathfrak{n}]$ and $\mathfrak{n}\subset\mathfrak{p}$, $[\mathfrak{n},\mathfrak{n}]\subset\mathfrak{k}$ we deduce that $\mathfrak{g}'\cap\mathfrak{p} = \mathfrak{n}$ which implies $D_e\pi(\mathfrak{n}) = T_o M'$. Let $v\in T_o M'$ and $X\in\mathfrak{n}$ with $D_e\pi(X) = v$. Then (Thm II.39) $t\longmapsto \exp(tX)_* o$ is the geodesic through (o,v). On the other hand $tX\in\mathfrak{g}'$ $\forall t$, hence $\exp(tX)\in G'$ $\forall t\in\mathbb{R}$ which implies $\exp(tX)_* o \in G'_* o = M'$ $\forall t\in\mathbb{R}$ and hence M' is geodesic at o. Now G' acts by isometries on M and transitively on M' which implies that M' is geodesic at every point and hence totally geodesic.

(2) Now $N \subset M$ is totally geodesic and $o \in N$. Let

$$\mathfrak{n} := (D_e\pi)^{-1}(T_oN)$$

and $\mathrm{Exp} = \mathrm{Exp}_o \circ D_e\pi|_{\mathfrak{p}} : \mathfrak{p} \to M$

the map used in Corollary II.41.

Then $\forall\, X, Y \in \mathfrak{n}$, $\mathrm{Exp}(tX)$ and $\mathrm{Exp}(tY)$ are M-geodesics through o with tangent vectors $D_e\pi(X) \in T_oN$ and $D_e\pi(Y) \in T_oN$ respectively. As a result:

$$\mathrm{Exp}\, tX \text{ and } \mathrm{Exp}\, tY$$

are contained in N $\forall\, t \in \mathbb{R}$.

Now consider the restriction of Exp to $\mathfrak{n}$; since $\operatorname{Exp}(\mathfrak{n}) \subset N$ we have that $D_{tY} \operatorname{Exp}(\mathfrak{n}) \subset T_{\operatorname{Exp}(tY)}(N)$.

Now Corollary II.41 gives:

$$\left(D_{tY} \operatorname{Exp}\right)(X) = D_0 L_{\exp tY} \circ D_e \pi \left(\sum_{n=0}^{\infty} \frac{\operatorname{ad}(tY)^{2n}(X)}{(2n+1)!} \right)$$

that is:

$$\left(D_0 L_{\exp tY}\right)^{-1} \left(D_{tY} \operatorname{Exp}\right)(X)$$

$$= D_e \pi \left(\sum_{n=0}^{\infty} \frac{\operatorname{ad}(tY)^{2n}(X)}{(2n+1)!} \right).$$

But (see Prop. I.22) $D_0 L_{\exp tY} : T_0 M \longrightarrow$ $\longrightarrow T_{(\exp tY)_0} M$ implements the parallel transport along $t \mapsto \exp tY$

it follows from Prop. II.44 that

$$\sum_{n=0}^{\infty}{}' \frac{\operatorname{ad}(tY)^{2n}(X)}{(2n+1)!} \in \mathfrak{r}$$

$\forall X, Y \in \mathfrak{r}, \forall t \in \mathbb{R}.$

Taking second derivative in t and setting $t=0$ we get:

$$\operatorname{ad}(Y)^2(X) \in \mathfrak{r} \quad \forall X, Y \in \mathfrak{r}.$$

Now for $Y, Z \in \mathfrak{r}$ we have

$$\operatorname{ad}(Y+Z)^2 = \operatorname{ad}(Y)^2 + \operatorname{ad}(Y)\operatorname{ad}(Z) + \operatorname{ad}(Z)\operatorname{ad}(Y) + \operatorname{ad}(Z)^2.$$

Which implies

$$[Y,[Z,X]] + [Z,[Y,X]] \in \mathfrak{r} \quad \forall X, Y, Z \in \mathfrak{r}.$$

Now by Jacobi:

$$[Z,[Y,X]] + \underbrace{[X,[Z,Y]]}_{-[X,[Y,Z]]} + \underbrace{[Y,[X,Z]]}_{-[Y,[Z,X]]} = 0$$

which implies

$$\cancel{2[Y,[X,Z]] +}$$

$$2[Y,[Z,X]] + [X,[Y,Z]] \in \mathcal{R}. \qquad (*)$$

Exchanging the role of X and Y gives

$$2[X,[Z,Y]] + [Y,[X,Z]] \in \mathcal{R} \qquad (**)$$

Subtracting $(**)$ from $(*)$ gives:

$$3[Y,[Z,X]] + 3[X,[Y,Z]] \in \mathcal{R}$$

which equals

$$3[[X,Y],Z] \in \mathcal{R}$$

by Jacobi. $\blacksquare$