Symmetric Spaces F.26.5

18. 2. 2020.

Remarks on the litterature.

– Berger: "Panoramic View of Riemannian geometry"

Presents it from a historical viewpoint, including the Gromovian revolution started in the 80's.
Very good section (see 4.4) on the Riemann curv. tensor, its relation to the sectional curvature and various ways to think about these objects.

– Boothby & do Carmo are textbooks the latter one being short and geobasic.

Concerning Symmetric Spaces:

- S. Helgason: comprehensive, pedestrian treatment.
- P.S. Eberlein: presents a good summary of Helgason and a treatment of modern developments. (basically till end of ~~[illegible]~~ twentieth century).
- J. Maubon: first 4 pages are an excellent summary of the needed Riemannian geometry.
- M. Burger: Video of MSRI Lectures.
- A. Iozzi: Skript of a "Symm. Space" course she gave trying to present the basics of the theory using Helgason's book.

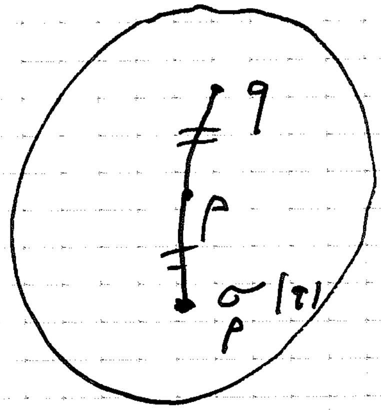
I. Introduction.

Let (M, g) be a Riemannian manifold. The Riemannian curvature tensor is an object that to every point $m \in M$ and tangent vectors $v, w \in T_m M$ associates an endomorphism

$$R_m(v, w) \in \mathrm{End}(T_m M)$$

of the tangent space at m. Looking for the most general notion of "constant curvature" Elie Cartan introduce in 1926 the notion of "Espace $\mathcal{E}$". Namely (M, g) is an "Espace $\mathcal{E}$" if the Riemannian curvature tensor is invariant under parallele transport; nowadays they are called locally symmetric spaces. It is only in subsequent papers, in 1929 that E. C. put an equivalent characterization in the foreground. Let $p \in M$ and $U \ni p$ be a normal neighborhood:

U



A geodesic symmetry at p is a map

$$s_p : U \to U$$

that fixes p and reverses any local geodesic through p.

Then M is an "Espace E" iff for every p there is a geodesic symmetry at p which is an isometry on its domain.

Now (M, g) connected is called globally symmetric if it is locally symmetric and in addition $s_p : M \to M$ is a globally ~~definite~~ defined isometry.

Now let (M, g) be connected, complete, locally symmetric space. Then its universal covering Riemannian manifold

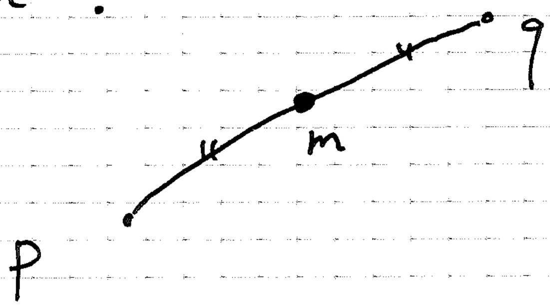
$(\tilde{M}, \tilde{g})$ has the same properties and is in addition simply connected.

The following is then fundamental:

<u>Thm I.1.</u> If (X, g) is a connected, simply connected, complete, locally symmetric space, it is globally symmetric.

Thus $(\tilde{M}, \tilde{g})$ is globally symmetric. Why is this fundamental? Because it establishes the connection with Lie group theory!

Let (X, g) be connected, complete, globally symmetric:



Then $s_m(p) = q$ and thus the group

$$Is(X) = \{ f : X \to X \text{ is an isometry} \}$$

acts transitively on X. ~~Th/ffkkk~~ ~~one can refine this argument and show that $G = Is($~~.

Now (X, d) with Riemannian distance is a metric space and one can endow $Is(X)$ with the compact open topology: it coincides with the topology of uniform convergence on compact sets. Thus $Is(X)$ is a topological group that is locally compact. In fact much more is true: let $K = \{g \in Is(X) : g(p) = p\}$ be the stabilizer of p in $Is(X)$.

Thm.$^{I.2}$ Let (X, g) be connected, complete, globally symmetric. Then $Is(X)$ admits a smooth structure which turns it into a Lie group; the transitive action

$$Is(X) \times X \longrightarrow X$$

is smooth

and the induced orbit map

$$Is(X)/_K \xrightarrow{\sim} X$$

is a diffeomorphism.

It is now time for a few examples.

Examples I. 3

(1) $\mathbb{E}^n = (\mathbb{R}^n, \langle\,,\,\rangle)$, euclidean n-space, is globally symmetric:

$$s_p(v) = 2p - v$$

which is clearly an isometry.

Let $o \in \mathbb{E}^n$ denote the origine.

$$Is(\mathbb{E}^n) = \mathbb{R}^n \rtimes O(n)$$

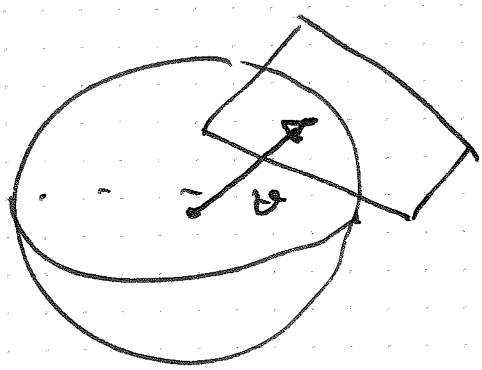
where $\mathbb{R}^n$ is the subgroup consisting of translations and $O(n)$ the orthogonal group, equivalently, the group of linear isometries.

It has constant sectional curvature $\equiv 0$.

(2) $\mathbb{S}^n = \{ x \in \mathbb{E}^{n+1} : d(x, o) = 1 \}$

with the Riemannian metric induced by $\mathbb{E}^{n+1}$.

$\mathbb{S}^2$



Now v determines an orthogonal dec.

$$\mathbb{R}v^{n+1} + (\mathbb{R}v^{n+1})^{\perp} = \mathbb{R}^{n+1}$$

Then $S_v(tv + y) = tv - y$ is an element of $O(n+1)$ whose restriction to $\mathbb{S}^n$ is the symmetry about v.

Observe that $S_v|_{\mathbb{S}^n}$ has two fixed points, v and $-v$.

One verifies $Is(\mathbb{S}^n) = \{ g|_{\mathbb{S}^n} : g \in O(n+1) \}$.

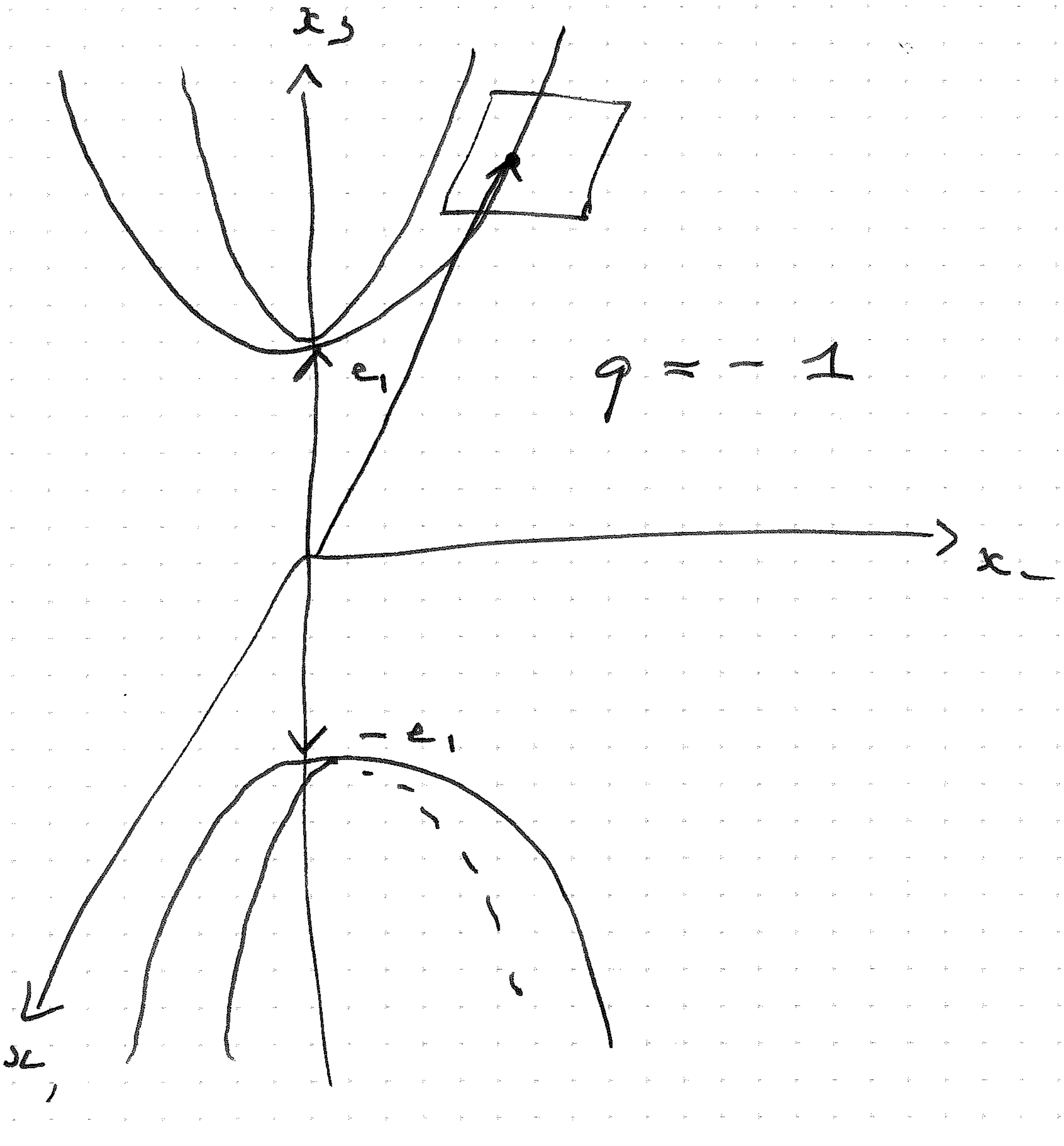
$\mathbb{S}^n$ has constant sectional curvature 1.

(3) $\mathbb{H}^n$: the hyperbolic n-space. Consider on $\mathbb{R}^{n+1}$ the standard form of signature $(n,1)$:

$$\langle x, y\rangle := \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}.$$

and $q(x) = \langle x, x\rangle$. Let

$$\mathbb{H}^n = \{ x \in \mathbb{R}^{n+1} : x_{n+1} \geq 1,\ q(x) = -1 \}$$



For every $x \in \mathbb{H}^n$, we have

$$\mathbb{R}^{n+1} = \mathbb{R}x \oplus (\mathbb{R}x)^\perp$$

where $(\mathbb{R}x)^\perp = \{y \in \mathbb{R}^{n+1} : \langle y, x \rangle = 0\}$ is also the tangent space of $\mathbb{H}^n$ at x. Since $q(x) = -1$, the restriction of $\langle\ ,\ \rangle$ to $(\mathbb{R}x)^\perp$ is positive definite and this gives a Riemannian metric on $\mathbb{H}^n$.

Let $O(n,1) = \{g \in GL(n+1, \mathbb{R}) : g$ preserves $\langle\ \ \rangle\}$

Then one can show that

$$Is(\mathbb{H}^n) = \{g|_{\mathbb{H}^n} : g \in O(n,1) : g(\mathbb{H}^n) = \mathbb{H}^n\}.$$

(exercise).

For $v \in \mathbb{H}^n$, $f_v(y) := -2v\langle v, y\rangle - y$ is in $O(n,1)$, preserves $\mathbb{H}^n$ and its restriction to $\mathbb{H}^n$ is the geodesic symmetry at v.

This space has constant sectional curvature $= -1$.

In fact there is a theorem that classifies all constant curvature simply connected manifolds, namely:

<u>Thm. I.4</u> If (X^n, g) is complete, s.c., with constant sectional curvature $K \in \{-1, 0, 1\}$:

$K = 1 \quad X^n \simeq \mathbb{S}^n$

$K = 0 \quad X^n \simeq \mathbb{E}^n$

$K = -1 \quad X^n \simeq \mathbb{H}^n$.

In fact, as we will see, the theory is much richer, for instance:

<u>Example I.5</u>: Let $\mathcal{P}^1(n) = \{A \in M_{n,n}(\mathbb{R}) :$ $\det A = 1$, A symmetric, A is positive definite$\}$

Now $SL(n,\mathbb{R})$ acts on $\mathcal{P}^1(n)$ as follows:

$$g_* A := g A \, {}^t g$$

Recall that if A is seen as a symmetric bilinear form this corresponds to the "change of variables" by ${}^t g$. From this one deduces that $SL(n,\mathbb{R})$ acts transi– on $\mathcal{P}'(n)$. One can use this to show that there is a structure of symmetric space on $\mathcal{P}'(n)$ for which

$$Is(\mathcal{P}'(n))^\circ = PSL(n,\mathbb{R}) := SL(n,\mathbb{R}) / \{\pm Id\}.$$

It is only for $n = 2$ that we find a coincidence with the previous list.

<u>Example I.6</u> In dimension 2. The list of (globally) symmetric spaces is

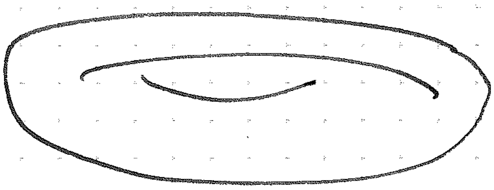
$$\mathbb{S}^2, \ \mathbb{E}^2, \ \mathbb{H}^2.$$

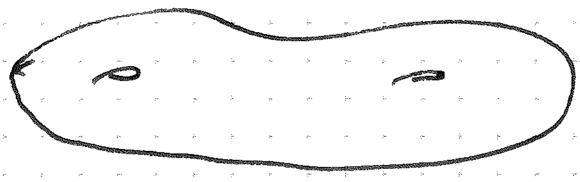
What about the (complete) locally symmetric ones. Let us restrict to the compact orientable ones. Here we can now turn things around and use: 20.2.2020.

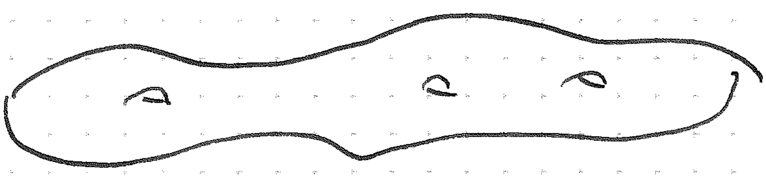
Thm I.7 Let S be a compact, connected, orientable smooth surface. Then ~~S is diffeomorphic to one of the following:~~ there is an ~~numerical~~ integer invariant, the genus $g(S) \in \{0, 1, 2, \ldots\}$ which determines the diffeomorphism type of S:

$g(S)$

$S^2 =$  0

$\mathbb{T}^2 =$  1

higher genus:  2

 3

And now the question is, which ones admit a locally symmetric Riemannian metric? Observe that it will necessarily be of constant curvature;

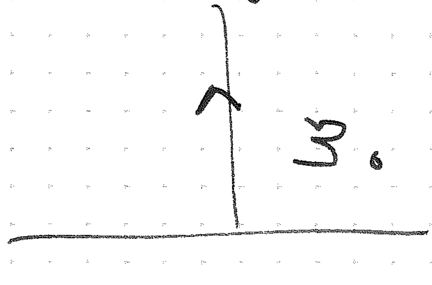
If (S, g) is a surface as above and g a locally symmetric metric then:

(1) $g(S)=0$, $S \cong S^2$ (isometric).

(2) $g(S)=1$, $S = \Gamma \backslash \mathbb{E}^2$ where $\Gamma < \mathbb{R}^2$ is a group of translations acting properly discontinuously without fixed points on $\mathbb{E}^2$.

For $g(S) \geq 2$ we look at the following more familiar model of $\mathbb{H}^2$ namely

$$\mathcal{H}^+ = \{ z \in \mathbb{C} : y > 0 \}$$



with Riemannian metric:

$$ds^2 = \frac{(dx)^2 + (dy)^2}{y^2}$$

Then $SL(2,\mathbb{R})$ acts on $\mathcal{H}^+$ by

$$g_* z = \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$$

and $PSL(2,\mathbb{R}) := SL(2,\mathbb{R}) / \pm Id$ acts effectively. Then $\mathcal{H}^+ \cong \mathbb{H}^2$ and

$Is(\mathcal{H}^+)^+$ = group of orientation pres. isometries is $PSL(2,\mathbb{R})$.

If now $g(S) \geq 2$ then

$$(S, g) \cong \Gamma \backslash \mathcal{H}^+$$

where $\Gamma < PSL(2,\mathbb{R})$ is a subgroup acting properly discont. without fixed points on $\mathcal{H}^+$.

So how many locally symmetric structures are there on a surface of genus $g \geq 2$? Assuming a proper formulation, one can show that this set has a manifold structure of dimension $6g - 6$; it is called Teichmüller space of genus g.

Coming back to our general introduction, we have seen that there is a relation between globally symmetric spaces and Lie group, via their group of isometries. In fact one of our first goals will be to use this to establish a Lie theoretic characterization which grosso modo goes as follows:

Let G be a connected Lie group and $\sigma : G \to G$ an involutive automorphism.

So $\sigma^2 = \mathrm{Id}$, σ is a smooth group automorphism. Assume:

$$G^\sigma := \{ g \in G : \sigma(g) = g \} \quad \text{is}$$

compact. Let

$$(G^\sigma)^\circ < K < G^\sigma .$$

Then the manifold G/K can be endowed with a G-inv. Riemannian metric making it a symmetric space. The point is: all symmetric spaces are obtained this way.

<u>Example I.6</u>: Illustrates that this is a powerfull way to construct symmetric spaces. Take $G = SL(n,\mathbb{R})$ and $\sigma(g) = {}^t g^{-1}$. Then σ is an involutive automorphism and

$$G^\sigma = \{ g \in SL(n,\mathbb{R}) : g = {}^t g^{-1} \} = SO(n).$$

Thus $SL(n,\mathbb{R})/SO(n) \simeq P^1(n)$ is a symm. space

This Lie theoretic characterization will allow us to express the Riemannian objects like parallel transport, exponential map, riemannian curvature tensor in terms of Lie theoretic objects.

We will then go one step further and consider $\mathfrak{g} = Lie(G)$, $\Theta := d\sigma : \mathfrak{g} \to \mathfrak{g}$.

As a linear involution

$$\mathfrak{g} = \underbrace{\mathfrak{k}}_{\substack{+1 \\ \text{eigensp.}}} \oplus \underbrace{\mathfrak{m}}_{-1 \text{ eigenspace}}$$

and $\mathfrak{k} = Lie(K)$.

In this way we will reduce the study of symmetric spaces to the study of orthogonal symmetric Lie algebras, that are pairs $(\mathfrak{g}, \Theta)$ consisting of a Lie Algebra and an involutive automorphism Θ

satisfying certain axioms.