III.3. Decomposition into irreducible symmetric spaces.

Our aim is to push the decomposition theorem a little further until we reach "irreducible pieces". Some care has to be taken since for instance euclidean space $\mathbb{E}^n$, $n \geq 2$, is a riemannian product in many ways. With the proper concepts we will however be able to obtain a canonical decomposition of so called reduced OSL's.

<u>Def. III.20</u> An OSL $(\mathfrak{g}, \theta)$ is <u>reduced</u> if $\mathfrak{u}$ does not contain any non-zero ideal of $\mathfrak{g}$.

Clearly since any subalgebra of the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is an ideal, if $(\mathfrak{g}, \theta)$ is reduced, we must have $Z(\mathfrak{g}) \cap \mathfrak{u} = (0)$ and hence $(\mathfrak{g}, \theta)$ is effective.

<u>Def III.21</u> An OSL $(\mathfrak{g}, \theta)$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is <u>irreducible</u> if

(1) $\mathfrak{g}$ is semisimple and $(\mathfrak{g}, \theta)$ is reduced.

(2) the Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ acts irreducibly on $\mathfrak{p}$.

With these definitions we have:

Theorem III.22

Let $(\mathfrak{g}, \theta)$ be a reduced OSL. Then $(\mathfrak{g}, \theta)$ is the direct product of a reduced OSL $(\mathfrak{g}_0, \theta_0)$ of euclidean type and irreducible OSL's $(\mathfrak{g}_i, \theta_i)$ $1 \leq i \leq a$. This decomposition is unique up to orders of the factors $1 \leq i \leq a$.

The proof (left as an exercise) uses the same overall strategy than the proof of Thm III.9 with an additional input:

Thm III.23 Let $K < GL(V)$ be a compact subgroup where V is a finite dimensional real vector space. Then there is a decomposition $V = \bigoplus_{i \in I} V_i$

into a direct sum of K-invariant

irreducible subspaces.

The idea of proof goes as follows:

first one uses a Haar measure on K

to produce a K-invariant scalar product

on V. Then one observes that if $W \subset V$

is K-invariant so is its orthogonal $W^\perp$.

Finally one chooses $W \subset V$ minimal invariant

and proceeds by induction on dim V.

Now with the notations of the proof of

Thm III. 9 namely

$$B_g(x, y) = \langle Ax, y \rangle \quad \forall x, y \in e$$

let $e = \bigoplus_{i=0}^{r} g_i$ be the direct sum decomposition into eigenspaces of A:

$A|_{q_i} = c_i \, Id$, $c_0 = 0$, $c_1 \neq 0, \dots, c_r \neq 0$

and all the c_i's are distinct.

Then $ad_g(\mathfrak{k})$ leaves each q_i invariant

and since $ad_g(u) = $ Lie U with

U compact we can apply Thm III. 23

and there exists therefore a direct

sum decomposition of $\bigoplus_{i=1}^{r} q_i$ into

subspaces p_i, $i = 1, \dots, a$ each of

which is contained in some q_j, which

are mutually orthogonal wrt B_g and

$ad_g u$ - minimal invariant. Then one

define $g_i = [p_i, p_i] + p_i$

$i = 1, \dots, a$

Then one shows:

(1) $\mathfrak{g}_i$ is a Θ-stable ideal in $\mathfrak{g}$.

(2) $B_{\mathfrak{g}_i} = B_{\mathfrak{g}}|_{\mathfrak{g}_i \times \mathfrak{g}_i}$ is non-degenerate.

Define then $\mathfrak{m} := \bigoplus_{i=1}^{a} \mathfrak{g}_i$. Then $\mathfrak{m}$ is a semisimple Θ-stable ideal in $\mathfrak{g}$

(3) $\mathfrak{g}_0 := Z_{\mathfrak{g}}(\mathfrak{m})$ the centralizer of $\mathfrak{m}$ in $\mathfrak{g}$ is a Θ-stable euclidean OSL.

Now we turn to the corresponding notions for RSP's and riemannian symmetric spaces.

Def. III.24 A riemannian symmetric pair (G, K) is

– reduced if the corresponding OSL is

– irreducible if the corresponding OSL is.

We observe that (G, K) is reduced iff K doesn't contain any connected non-trivial normal subgroup of G or in other words:

(III.25) The kernel N of the G-action on G/K is discrete.

We conclude that if M is RSS and $G = Is(M)^\circ$, $K = Stab_G(o)$ the associated RSP, then (G, K) is reduced.

~~We define:~~ It thus makes sense to define:

Def III.26 A RSS M is irreducible if the corresponding RSP is.

From the above we deduce

(III.27) A riemannian symmetric space M is irreducible iff for $G = Is(M)^\circ$, $K = Stab_G(o)$ the Lie algebra $\mathfrak{g}$ of G is semisimple and the representation $Ad: K^{\circ} \to GL(\mathfrak{p})$ is irreducible, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition.

As in Thm III. 19, one can globalise the decomposition given by Thm III. 22 and obtain

<u>Corollary III. 28.</u>

A riemannian symmetric space M is isometric to a riemannian product

$$M_0 \times M_1 \times \cdots \times M_a$$

where $M_0 = E^n$, and $M_1, \ldots, M_a$ are irreducible. This product decomposition is unique up to permutation of the factors $M_1, \ldots, M_a$.

Irreducible riemannian symmetric pairs (hence spaces) satisfy an interesting dichotomy:

<u>Prop. III.29</u> Let (G, K) be an irreducible riemannian symmetric pair $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition and $B_{\mathfrak{g}}$ the Killing form of $\mathfrak{g}$. Then there is a (up to scaling) unique G-invariant riemannian metric on G/K and one of the following holds:

(1) $B_{\mathfrak{g}}\big|_{\mathfrak{p}\times\mathfrak{p}} \gg 0$ and G/K is of non-compact type.

(2) $B_{\mathfrak{g}}\big|_{\mathfrak{p}\times\mathfrak{p}} \ll 0$ and G/K is of compact type.

In the first case $B_{\mathfrak{g}}\big|_{\mathfrak{p}\times\mathfrak{p}}$ gives rise to a G-invariant riemannian metric, and in the second case it is $-B_{\mathfrak{g}}\big|_{\mathfrak{p}\times\mathfrak{p}}$.

<u>Proof</u>: As usual, let $\langle\ \rangle$ be any $Ad\,K$-invariant scalar product on $\mathfrak{p}$ and $A \in \mathrm{End}\,\mathfrak{p}$ defined by

$$B_g(x,y) = \langle Ax, y\rangle \quad \forall x,y \in \mathfrak{p}.$$

~~Since A and Ad(K) commute~~

Recall that A is symmetric hence diagon. over $\mathbb{R}$. Since $Ad(K)$ and A commute any eigenspace of A is $Ad(K)$-invariant which since $Ad(K)$ acts irreducibly on $\mathfrak{p}$ implies $A = \lambda\,\mathrm{Id}$ for some $\lambda \neq 0$. Thus $B_g(x,y) = \lambda\langle x,y\rangle$ which shows the uniqueness statement. The sign of λ then determines whether we are in case (1) or (2).

$\boxed{S}$

IV. On some global properties of symmetric spaces. Duality.

In the first subsection we indicate how the distinction between compact type, non-compact type and euclidean type is reflected in the sign of the sectional curvature. With a theorem of Hadamard we conclude then that a symmetric space of non-compact type is diffeomorphic to $\mathbb{R}^n$ and is a CAT(0)-space. In the second subsection we establish some basic properties of semisimple groups and use them to show that if (G, K) is a riemannian symmetric pair of compact type, then G is compact and the associated symmetric space $M = G/K$ is as well.

The last subsection is devoted to the duality between symmetric spaces of compact and non-compact type.

IV. 1. Curvature of symmetric spaces.

Recall that if (M, g) is a riemannian manifold and ∇ the Levi Civita connection, the multilinear mapping

$$R : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \times \mathrm{Vect}(M) \longrightarrow \mathrm{Vect}(M)$$

defined by:

$$R(X,Y)Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$$

is a tensor, that is, the value $\left(R(X,Y)Z\right)_p$ at a point p only depends on the values X_p, Y_p, Z_p.

A related curvature notion which is equivalent to the knowledge of the riemannian curvature tensor is the sectional curvature σ_p defined for every $p \in M$ on the grassmannian of 2-planes in T_pM:

$$\sigma_p : Gr_2(T_pM) \longrightarrow \mathbb{R}$$

$$\sigma_p(P) = -\langle R(u,v)u, v\rangle$$

where $\{u,v\}$ is an orthonormal basis of the 2-plane $P \subset T_pM$.

We refer to Berger's book "A panorama of Riemannian Geometry" Section 4.4 for a discussion of various aspects of the riemannian curvature tensor.

A fundamental theorem due to J. Hadamard gives global metric information for riemannian manifolds with sectional curvature ≤ 0.

<u>Thm IV. 1.</u> Let M be a complete riemannian manifold with $\sigma_p(P) \leq 0$ $\forall p \in M$, $\forall P \in G_2(T_pM)$.

Given $p \in M$ and $v \in T_pM$ and identifying $T_v(T_pM)$ with T_pM we have: $D_v \exp_p : T_pM \longrightarrow T_{\exp_p(v)}(M)$

is norm increasing, that is:

$$\| D_v(\exp_p)(\xi) \| \geq \| \xi \| \quad \forall \xi \in T_p M.$$

In particular

(1) length$(\sigma) \leq$ length$(\exp_p \circ \sigma)$

for any rectifiable path $\sigma : [0,1] \to T_p M$.

(2) $\exp_p : T_p M \to M$ is a covering map.

(3) If M is simply connected, $\exp_p$ is a diffeomorphism and

$$d(\exp_p(v), \exp_p(w)) \geq \| v - w \|$$

$\forall v, w \in T_p M$.

~~We will return~~.

In this subsection we want to discuss the following:

Thm IV. 2

Let (G, K) be a RSP and M the associated symmetric space equipped with a G-invariant riemannian metric.

(1) If (G, K) is of compact type the sectional curvature is everywhere ≥ 0.

(2) If (G, K) is of non compact type the sectional curvature is everywhere ≤ 0.

(3) If (G, K) is of euclidean type the sectional curvature $\equiv 0$.

~~The proof of~~

Thus Hadamard's theorem applies to symmetric spaces of non-compact type.

The proof of Thm IV.2 is based on a (non-trivial) computation which we will skip.

See Helgason Thm IV 4.2 p. 215.

Thm IV.3

Let (G, K) be a RSP, M the associated symmetric space and R the riemannian curvature tensor wrt any G-invariant metric. Let as usual

$$D_e\pi : \mathfrak{p} \longrightarrow T_o M$$

be the canonical isomorphism. Then

$\forall\, X, Y, Z \in \mathfrak{p}$:

$$R_o(\bar{X}, \bar{Y})\bar{Z} = -\overline{[[X,Y],Z]}$$

where we set $\bar{T} = D_e\pi(T)$, $T \in \mathfrak{p}$.

Proof of Thm IV. 2 :

We use IV. 3 to compute the sectional curvature in each case.

Let $(\mathfrak{g}, \Theta)$ be the OSL associated to (G, K), $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition and $B_{\mathfrak{g}}$ the Killing form. Then we have $\forall\, X_1, X_2 \in \mathfrak{p}$:

$$B_{\mathfrak{g}}\left(-[[X_1, X_2], X_1], X_2\right)$$

$$= B_{\mathfrak{g}}\left([X_1, [X_1, X_2]], X_2\right)$$

$$= - B_{\mathfrak{g}}\left([X_1, X_2], [X_1, X_2]\right).$$

If (G, K) is of compact type we may take $-B_{\mathfrak{g}}\big|_{\mathfrak{p} \times \mathfrak{p}}$ as riemannian metric at o, under the identific.