$D_e \pi : \mathfrak{p} \longrightarrow T_o M$.

In this case if $X_1, X_2 \in \mathfrak{p}$ are such that $\bar{X}_1, \bar{X}_2$ are orthonormal in $T_o M$, we have:

$$\sigma_o(\langle \bar{X}_1, \bar{X}_2 \rangle) = -\langle R_o(\bar{X}_1, \bar{X}_2) \bar{X}_1, \bar{X}_2 \rangle$$

$$= \langle \overline{[[X_1, X_2], X_1]}, \bar{X}_2 \rangle$$

$$= -B_{\mathfrak{g}}([[X_1, X_2], X_1], X_2)$$

$$= -B_{\mathfrak{g}}([X_1, X_2], [X_1, X_2]) = \| \overline{[X_1, X_2]} \|^2$$

$$\geq 0.$$

In the case $(G, K)$ is of non-compact type we can take $B_{\mathfrak{g}}|_{\mathfrak{p} \times \mathfrak{p}}$ as riemannian metric at $o$ and the computation above leads to

$$\sigma_0(\langle \bar{X}_1, \bar{X}_2 \rangle) = - \| \overline{[X_1, X_2]} \|^2 \leq 0.$$

The assertion about the condition type is clear.

## IV. 2. Semisimple Lie groups : basics.

We have seen that the decomposition theorem of symmetric spaces leads to a class of Lie algebras, the semisimple ones. The corresponding Lie groups are then closely related to RSS and in this short introduction we will establish an important global property of those with negative definite Killing form.

<u>Def. IV. 4</u> A Lie group is semisimple if its Lie algebra is.

Beside their importance for the study of RSS, semisimple groups are together with solvable ones the building blocs of general Lie groups.

Next let us indicate where the terminology semisimple comes from.

<u>Def. IV. 5</u> A Lie algebra is simple if it is non-abelian and contains no proper non-zero ideal.

Then:

<u>Thm IV. 6.</u> (Cartan's Criterion) A (real) Lie algebra $\mathfrak{g}$ is semisimple if and only if it is sum of simple ideals.

In fact this theorem is valid for Lie algebras over a field of characteristic zero.

One direction is an elegant ~~theo~~ argument of Dieudonné. Assume $K_{\mathfrak{g}}$ is non-degenerate. First we observe that $\mathfrak{g}$ has no nonzero ~~ideal~~ abelian ideal. Indeed if $\mathfrak{a} \triangleleft \mathfrak{g}$ is an abelian ideal then one verifies easily that

$$(\mathrm{ad}(A)\,\mathrm{ad}(X))^2 = 0 \qquad \forall A \in \mathfrak{a}, \forall X \in \mathfrak{g}$$

and hence

$$K_{\mathfrak{g}}(A, X) = \mathrm{tr}(\mathrm{ad}(A)\,\mathrm{ad}(X)) = 0 .$$

Next let $\mathfrak{m} \triangleleft \mathfrak{g}$ be a nonzero ideal of minimal dimension. Then $[\mathfrak{m}, \mathfrak{m}]$ being an ideal in $\mathfrak{g}$ and $\mathfrak{m}$ being not abelian implies $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$. Let $\mathfrak{m}^{\perp}$ be the subspace of $\mathfrak{g}$ orthogonal to $\mathfrak{m}$ wrt the Killing form. Then one verifies that $\mathfrak{m}^{\perp}$ is an ideal. Thus either $\mathfrak{m} \cap \mathfrak{m}^{\perp} = (0)$ or $\mathfrak{m} \subset \mathfrak{m}^{\perp}$. The latter case cannot occur: let $A = \sum_i \underbrace{[B_i, C_i] \in \mathfrak{m}}_{\text{with } B_i, C_i \in \mathfrak{m}}$ and $X \in \mathfrak{g}$. Then:

$$K_g(A, X) = -\sum_i K_g\Big(\underset{\mathfrak{m}}{\underset{\cap}{C_i}}, \underset{\mathfrak{m}}{\underset{\cap}{[B_i, X]}}\Big)$$

$$= 0 \quad \text{since } \mathfrak{m} \subset \mathfrak{m}^{\perp}.$$

But this contradicts the assumption that $K_g$ is not degenerate. Thus $\mathfrak{m} \cap \mathfrak{m}^{\perp}$ and $\mathfrak{g}$ is the direct sum of the ideals $\mathfrak{m}$ and $\mathfrak{m}^{\perp}$.

The other direction uses Cartan's criterion for solvability of a Lie algebra $\mathfrak{g}$ (in characteristic zero):

Thm. IV.7 (Cartan's Criterion)

$\mathfrak{g}$ is solvable iff $K_{\mathfrak{g}}(\mathfrak{g}, [\mathfrak{g},\mathfrak{g}]) = 0$.

Now we turn to a property of the adjoint representation of a semisimple Lie group.

~~namely~~

Let $G$ be a real connected Lie group with Lie algebra $\mathfrak{g}$, $Ad : G \to GL(\mathfrak{g})$ its adjoint representation. Then $Ad(G)$ is contained in $Aut(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$, which is a closed subgroup of $GL(\mathfrak{g})$. In fact since $G$ is connected we have $Ad(G) \subset (\mathrm{Aut}\,\mathfrak{g})^0$.

When $G$ is connected semisimple we have:

<u>Thm IV.8</u> If $G$ is connected semisimple we have $Ad(G) = (\mathrm{Aut}\,\mathfrak{g})^0$.

The proof uses the analog on the level of Lie algebras:

<u>Prop. IV.9</u>. If $\mathfrak{g}$ is semisimple then

$$\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$$

in other words every derivation of $\mathfrak{g}$ is inner.

<u>Proof</u>: We have the following inclusion of spaces: $\mathrm{ad}(\mathfrak{g}) \subset \mathrm{Der}\,\mathfrak{g} \subset \mathrm{End}\,\mathfrak{g}$.

On $\mathrm{End}(\mathfrak{g})$, $(A, B) \mapsto \mathrm{tr}\, AB$ defines a <u>non-degenerate</u> symmetric bilinear form. Its restriction to $\mathrm{ad}(\mathfrak{g})$ is non-degenerate since $\mathfrak{g}$ is semisimple.

Thus it suffices to show that if $D \in \mathrm{Der}\, \mathfrak{g}$ is orthogonal to $\mathrm{ad}(\mathfrak{g})$ then it vanishes.

Now we compute $\forall\, x, Y \in \mathfrak{g}$:

$$K_{\mathfrak{g}}(Dx, Y) = \mathrm{tr}(\mathrm{ad}(Dx)\, \mathrm{ad}(Y))$$

But $\quad \mathrm{ad}(Dx)(Z) = [Dx, Z]$

$$= D([x, Z]) - [x, DZ]$$

that is, $\quad \mathrm{ad}\, Dx = [D, \mathrm{ad}(x)]$.

Hence

$$K_{\mathfrak{g}}(Dx, Y) = \mathrm{tr}([D, \mathrm{ad}(x)]\, \mathrm{ad}(Y))$$

$$= tr\left( D\,ad(X)\,ad(Y) - ad(X)\,D\,ad(Y) \right)$$

$$= tr\left( D\,ad(X)\,ad(Y) - D\,ad(Y)\,ad(X) \right)$$

$$= tr\left( D\left[ad(X), ad(Y)\right]\right)$$

$$= tr\left( D\,ad([X,Y])\right) = 0 \qquad \forall\, X,Y \in \mathfrak{g}.$$

Hence $B_{\mathfrak{g}}(DX, Y) = 0 \quad \forall\, X,Y \in \mathfrak{g}$

which implies $D = 0$. $\boxed{\xi}$

Here is a lemma left as exercise:

<u>Lemma IV.10</u> Let $\mathfrak{g}$ be a real Lie algebra. Then $Aut(\mathfrak{g})$ is a Lie subgroup of $GL(\mathfrak{g})$ and its Lie algebra coincides with $Der(\mathfrak{g})$.

The proof of Thm IV.8 is then straight-forward. Indeed, we have that

$$\mathrm{Lie}\, \mathrm{Ad}(G) = \mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g}) = \mathrm{Lie}(\mathrm{Aut}\,\mathfrak{g})$$

which shows that $\mathrm{Ad}(G) \subset \mathrm{Aut}\,\mathfrak{g}$ have the same Lie algebras; since $G$, hence $\mathrm{Ad}(G)$ is connected, this implies $\mathrm{Ad}(G) = \mathrm{Aut}(\mathfrak{g})^\circ$.

Now we are in a position to prove

<u>Thm IV.11</u> If $G$ is a connected Lie group whose Lie algebra has a negative definite Killing form then $G$ is compact.

<u>Proof</u>: Let $\mathfrak{g} = \mathrm{Lie}(G)$. Then $G$ is semi-simple and hence by Thm IV.8

$$\mathrm{Ad}(G) = \mathrm{Aut}(\mathfrak{g})^\circ.$$

Now $Aut(\mathfrak{g}) < GL(\mathfrak{g})$ is a closed subgroup preserving $B_{\mathfrak{g}} \ll 0$ hence it is compact and so is $Ad(G)$.

Now we consider $G \times G$ with ~~involution~~ automorphism $\sigma(g,h) = (h,g)$ which is an involution whose fixed point subgroup

$$(G \times G)^{\sigma} = \{(g,g) : g \in G\} := \Delta(G)$$

is the diagonal subgroup of $G \times G$.

Consider the adjoint representation

$$Ad_{G\times G} : G \times G \longrightarrow GL(\mathfrak{g} \times \mathfrak{g}).$$

The image $Ad_{G\times G}(\Delta(G))$ coincides with

$$\{(A,A) \in GL(\mathfrak{g} \times \mathfrak{g}) : A \in Aut(\mathfrak{g})^{\circ}\}$$

and is hence compact. Thus

$(G \times G, \Delta(G))$ is a riemannian symmetric pair. We consider the $G \times G$ - equivariant diffeomorphism

$$\varphi : G \times G / \Delta(G) \longrightarrow G$$

$$(g, h) \cdot \Delta(G) \longmapsto g \cdot h^{-1}$$

where the $G \times G$ action on $G$ is given by

$$(g, h) x = g x h^{-1}.$$

We fix a $(G \times G)$ - invariant riemannian metric on $G \times G / \Delta(G)$; via $\varphi$ it gives a $(G \times G)$ - invariant riemannian metric on $G$, or simply put, a left and right invariant riemannian metric on $G$.

Let $\mathfrak{p} = \{ (X, -X) : X \in \mathfrak{g} \}$. Then the geodesics through $e\Delta$ in $G \times G / \Delta$

are the $(\exp tX, \exp(-tX))$ orbit of $\Delta$ and under $D$, we conclude that the geodesics on $G$ through $e$ for the biinvariant metric are given by

$$t \mapsto \exp(2tX) \quad , \quad X \in \mathfrak{g}.$$

Now we proceed by contradiction. Assume $G$ is not compact. We use the following general, easy to prove fact:

If $(M,g)$ is a complete riemannian manifold and $M$ is not compact, then given $q \in M$ there exists a geodesic ray:

$$r : [0, \infty) \longrightarrow M,$$

$$r(0) = q, \quad d(r(t), r(0)) = t \quad \forall t \geq 0.$$

Thus if $G$ is not compact, denoting $d$ the biinvariant Riemannian distance on $G$, there exists $X \in \mathfrak{g}$ such that $d(\exp(2tX), e) = t$ $\forall t \geq 0$.

Now consider
$$\mathbb{R} \longrightarrow Ad(G)$$
$$t \longmapsto Ad(\exp 2tX)$$

whose closure $\mathbb{T} < Ad(G)$ is an abelian connected compact subgroup, hence a torus. By a theorem of Kronecker there exists $t_n \longrightarrow +\infty$ such that
$$\lim_{n \to \infty} Ad(\exp 2t_n X) = Id.$$

Since $Ker\, Ad = Z(G)$ there exists hence $z_n \in Z(G)$ with
$$\lim_{n \to \infty} d(\exp 2t_n X, z_n) = 0.$$

Now let $a \in G$ and consider

$$\eta(t) = a(\exp tX)a^{-1}$$

$$\gamma(t) = \exp tX .$$

Then: since $z_n \in Z(G)$ and $d$ is biinvariant:

$$d(\eta(t_n), z_n) = d(a(\exp t_n X)a^{-1}, z_n)$$

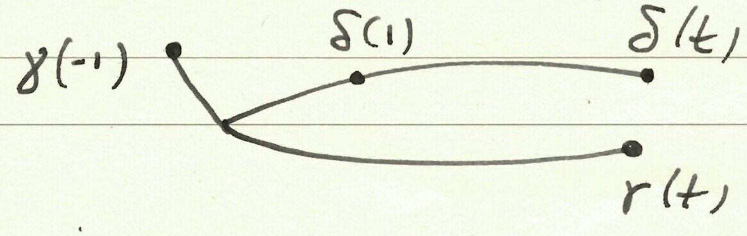
$$= d(\exp t_n X, \underbrace{a^{-1} z_n a}_{= z_n})$$

$$= d(\gamma(t_n), z_n) \xrightarrow[n \to \infty]{} 0$$

hence

$$\lim_{n \to \infty} d(\eta(t_n), \gamma(t_n)) = 0$$

Remark $\eta, \gamma$ are two geodesics through $e$. Assume now $\eta(t) \neq \gamma(t)$ for some $t > 0$.



then $\dot{\delta}(0) \neq \dot{\eta}(0)$. In particular:

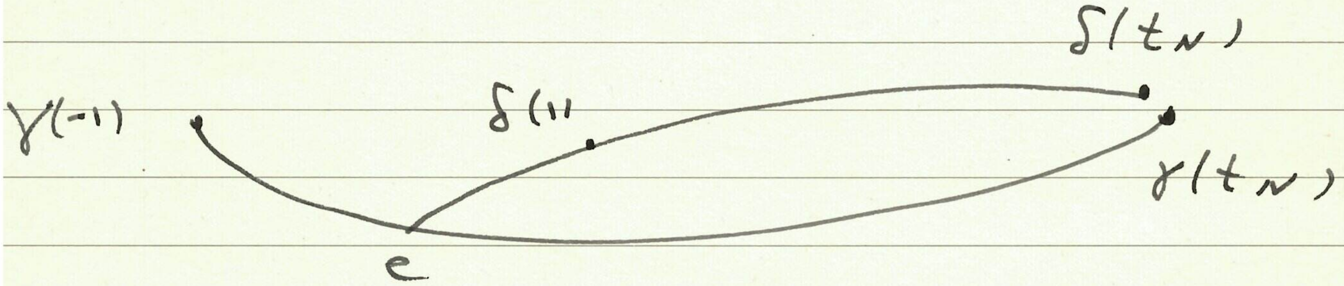
$$d(\gamma(-1), \delta(1)) < 2.$$

Otherwise ~~$d(\gamma(-1), \delta(1)$~~ the concatenation of $\gamma|_{[-1,0]} \cup \eta|_{[0,1]}$ would be a ~~distance~~ length minimizing path hence a geodesic.

Now pick $N$ with

$$d(\gamma(t_N), \delta(t_N)) < 2 - d(\gamma(-1), \delta(1)).$$

And consider the path:




consisting of a length min. geodesic from $\gamma(-1)$ to $\delta(1)$, the segment $\delta|_{[1,t_N]}$ and a length min. geodesic from $\delta(t_N)$ to $\gamma(t_N)$.

This path has length $< t_N + 1 = d(\gamma(-1), \gamma(t_N))$

which is a contradiction. Thus

$$\gamma(t) = \eta(t) \quad \forall t$$

that is $\exp(2t \, \mathrm{Ad}(a)(X)) = \exp(2tX), \forall t$

hence $\qquad \mathrm{Ad}(a)(X) = X \quad \forall a \in A.$

But this implies $\dim \mathfrak{z}(\mathfrak{g}) \geq 1$ contradicting

semisimplicity of $\mathfrak{g}$. $\boxed{\mathfrak{Z}}$