\[ D_{\alpha} \mathbf{u} : \mathbb{R} \rightarrow T_{\alpha}M. \]

In this case if \( x_1, x_2 \in \mathbb{R} \) are such that \( x_1, x_2 \) are orthonormal in \( T_{\alpha}M \),
we have:

\[
\sigma_0 \langle \bar{x}_1, \bar{x}_2 \rangle = -\langle R_0(x_1, x_2) \cdot \bar{x}_1, \bar{x}_2 \rangle
\]

\[
= \langle [\bar{x}_1, \bar{x}_2], x_1 \rangle, \bar{x}_2 \rangle
\]

\[
= -B_{\bar{x}_0} \langle [x_1, x_2], x_1 \rangle, x_2 \rangle \rangle
\]

\[
\leq -B_{\bar{x}_0} \langle [x_1, x_2], [x_1, x_2] \rangle = \| [x_1, x_2] \|^2
\]

\[ \geq 0. \]

In the case \((c, \mathbb{R})\) is of non-compact type we can take \( B_{\bar{x}_0} \) as

a Riemannian metric at \( 0 \) and the
The assertion about the quadratic form is clear.

IV. 2. Semisimple Lie groups: bases.

We have seen that the decomposition theorem of symmetric spaces leads to a class of Lie algebras, the semisimple ones. The corresponding Lie groups are then closely related to RGG and in this short introduction we will establish an important global property of those with negative definite Killing form.
Def. IV.4 A Lie group is semisimple if its Lie algebra is.

Beside their importance for the study of RFS, semisimple groups are together with soluble ones, the building blocks of general Lie groups.

Next let us indicate where the terminology semisimple comes from.

Def. IV.5 A Lie algebra is simple if it is non-nilpotent and contains no proper non-zero ideals.

Then:

Thm. IV.6. (Cartan's Criterion) A Lie algebra $g$ is semisimple if and only if it is sum of simple ideals.
In fact this theorem is valid for Lie algebras over a field of characteristic 0. One direction is an elegant argument of Dieudonné. Assume $K_g$ is non-degenerate. First we observe that $g$ has no nonzero nilpotent abelian ideal. Indeed if $\mathfrak{a} \subset \mathfrak{g}$ is an abelian ideal then one verifies easily that

$$\left( \text{ad}(A) \text{ad}(x) \right)^2 = 0 \quad \forall A \in \mathfrak{a}, \forall x \in \mathfrak{g}$$

and hence

$$K_g(A, x) = \text{tr} \left( \text{ad}(A) \text{ad}(x) \right) = 0.$$ 

Next let $\mathfrak{g}$ be a nonzero ideal of minimal dimension. Then $[\mathfrak{m}, \mathfrak{m}]$ being an ideal in $\mathfrak{g}$ and $\mathfrak{m}$ being not abelian
implies $M = [m, m]$. Let $m^+$ be
the subspace of $g$ orthogonal to $m$ wrt
the Killing form. Then one verifies that
$m^+$ is an ideal. Thus either $m \cap m^+ = \{0\}$
or $m \subset m^+$. The latter case cannot
occur: let $A = \sum [B_i, C_i] \in M$, and
with $B_i, C_i \in M$

$X \in g$. Then:

$$K_g(A, X) = - \sum_i K_g(C_i, [B_i, X])$$

$$= 0 \quad \text{since } M \subset m^+.$$

But this contradicts the assumption that
$K_g$ is not degenerate. Thus $m \cap m^+$
and $g$ is the direct sum of the ideals
$m$ and $m^+$. 
The other direction uses Cartan's criterion for solvability of a Lie algebra $\mathfrak{g}$ (in characteristic 0):

Thm. IV.7 (Cartan's Criterion)

$\mathfrak{g}$ is solvable if $\text{Ker} (\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

Now we turn to a property of the adjoint representation of a semisimple Lie group.

Let $G$ be a real connected Lie group with Lie algebra $\mathfrak{g}$, $\text{Ad} : G \to \text{GL}(\mathfrak{g})$ its adjoint representation. Then $\text{Ad}(G)$ is contained in $\text{Aut}(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$, which is a closed
subgroup of $GL(g)$. In fact since $G$ is connected we have $Ad(G)\subseteq (Aut_g)^\circ$.

When $G$ is connected semisimple we have:

Thm IV.8. If $G$ is connected semisimple we have $Ad(G) = (Aut_g)^\circ$.

The proof uses the analogy on the level of Lie algebras:

Prop. IV.9. If $g$ is semisimple then $Der(g) = ad(g)$

in other words every derivation of $g$ is inner.

Proof: We have the following inclusions of spaces: $ad(g) \subseteq Der(g) \subseteq End_g$. 
On $\text{End}(g)$, $(A, B) \mapsto \text{tr} A B$ defines a non-degenerate symmetric bilinear form. Its restriction to $\text{ad}(g)$ is non-degenerate since $g$ is semisimple. Thus it suffices to show that if $D \in \text{Der} g$ is orthogonal to $\text{ad}(g)$ then $D$ vanishes.

Now we compute $\forall x, y \in g$:

\[ K_g(Dx, y) = \text{tr} \left( \text{ad}(Dx) \text{ad}(y) \right) \]

But \[ \text{ad}(Dx)(z) = [Dx, z] \]

\[ = [D, [x, z]] - [x, Dz] \]

that is, \[ \text{ad} Dx = [D, \text{ad}(x)] \].

Hence

\[ K_g(Dx, y) = \text{tr} \left( [D, \text{ad}(x)] \text{ad}(y) \right) \]
\[-\text{tr} - \text{tr}\]

\[= \text{tr} (D \text{ad}(X) \text{ad}(Y) - \text{ad}(X) D \text{ad}(Y))\]

\[= \text{tr} (D \text{ad}(X) \text{ad}(Y) - D \text{ad}(Y) \text{ad}(X))\]

\[= \text{tr} (D [\text{ad}(X), \text{ad}(Y)])\]

\[= \text{tr} (D \text{ad}([X,Y])) = 0 \quad \forall X, Y \in g.\]

Hence \(\text{B}_g (D X, Y) = 0 \quad \forall X, Y \in g\)

which implies \(D = 0.\) \(\Box\)

Here is a lemma left as exercise:

**Lemma IV.10** Let \(g\) be a real Lie algebra. Then \(\text{Aut}(g)\) is a Lie subgroup of \(\text{GL}(g)\) and its Lie algebra coincides with \(\text{Der}(g).\)
The proof of Thm IV. 8 is then straightforward. Indeed, we have that
\[ \text{Lie } \text{Ad}(G) = \text{ad}(g) = \text{Der}(g) = \text{Lie}(\text{Aut}(g)) \]
which shows that \( \text{Ad}(G) \subset \text{Aut}(g) \) have the same Lie algebras. Since \( G \), hence \( \text{Ad}(G) \) is connected, this implies \( \text{Ad}(G) = \text{Aut}(g) \).

Now we are in a position to prove

**Thm IV. 11**  If \( G \) is a connected Lie group whose Lie algebra has a negative definite Killing form then \( G \) is compact.

**Proof:** Let \( g = \text{Lie}(G) \). Then \( G \) is semi-simple and hence by Thm IV. 8
\[ \text{Ad}(G) = \text{Aut}(g) \]
Now $\text{Aut}(G) \leq \text{GL}(G)$ is a closed subgroup preserving $B_G \leq G$ hence it is compact and so is $\text{Ad}(G)$ automorphism.

Now we consider $G \times G$ with convolution $\sigma(g, h) = (h, g)$. Which is an involutive whose fixed point subgroup

$$(G \times G)_{\sigma} = \{(g, g) : g \in G\} = \Delta(G)$$

is the diagonal subgroup of $G \times G$.

Consider the adjoint representation

$$\text{Ad}: G \times G \to \text{GL}(G \otimes G).$$

The image $\text{Ad}_{G \times G}(\Delta(G))$ coincides with

$$\{(A, t) \in \text{GL}(G \times G) : A \in \text{Aut}(G)\}$$

and is hence compact. Thus

$$(G \times G, \Delta(G))$$

is a riemannian.
symmetric pair. We consider the $6 \times 6$-equivariant diffeomorphism

$$6 \times 6 / \Delta(6) \rightarrow 6$$

$$(g, h) \cdot \Delta(6) \rightarrow g \cdot h^{-1}$$

where the $6 \times 6$ action on $6 \times 6$ given by

$$(g, h)x = gxh^{-1}.$$ 

We fix a $(6 \times 6)$-invariant Riemannian metric on $6 \times 6 / \Delta(6)$; it gives a $(6 \times 6)$-invariant Riemannian metric on $6$, or simply put, a left and right invariant Riemannian metric on $6$.

Let $\mathbf{p} = \{(x, -x) : x \in \mathbb{R}\}$. Then the geodesics through $e \Delta$ in $6 \times 6 / \Delta$
are the \((\exp t x, \exp(-t x))\) orbit of
\(\Delta\) orbit under \(D\), we conclude that
the geodesics on \(G\) through \(e\) for the
biinvariant metric are given by
\[ t \mapsto \exp(2t x) , \quad x \in \mathfrak{g}. \]
Now we proceed by contradiction. Assume
\(G\) is not compact. We use the following
general, easy to prove fact:

If \((M, g)\) is a complete riemannian manifold
and \(M\) is not compact, then given \(g \in \mathfrak{m}\)
there exists a geodesic ray:
\[ r : [0, \infty) \rightarrow M, \]
\[ r(0) = g , \quad d(r(t), r(s)) = t \quad \forall t, s \geq 0. \]
Thus if \( G \) is not compact, denoting the bi-invariant Riemannian distance on \( G \), there exists \( x \in g \) such that \( d(\exp(t x), e) = t \) \( \forall t > 0 \).

Now consider \( \mathbb{R} \to \text{Ad}(G) \)
\[
    t \mapsto \text{Ad}(\exp(2t x))
\]
whose closure \( \overline{T} \subset \text{Ad}(G) \) is an abelian connected compact subgroup, hence a torus. By a theorem of Kronecker there exists \( t_n \to +\infty \) such that
\[
    \lim_{n \to \infty} \text{Ad}(\exp(2t_n x)) = 1_{\text{Id}}.
\]
Since \( \text{Ker Ad} = Z(G) \) there exists hence
\[
    z_n \in Z(G) \text{ with }
    \lim_{n \to \infty} d(\exp(2t_n x), z_n) = 0.
\]
Now let \( a \in G \) and consider

\[
y(1) = a (\exp(2t^2))^{-1}, \\
y(1) = \exp(2t^2).
\]

Then, since \( 3_n \in \mathbb{Z}(G) \) and \( d \) is bi-invariant:

\[
d(y(1), 3_n) = d(a(\exp(2t^2))^{-1}, 3_n)
\]

\[
= d(\exp(2t^2), a^{-1}3_n)
\]

\[
= d(3_n, 3_n) \xrightarrow{n \to \infty} 0
\]

hence

\[
\lim_{n \to \infty} d(y(tn), y(tn+1)) = 0
\]

Remark: \( y, \gamma \) are two geodesics through \( e \). Assume now \( y(1) \neq \gamma(1) \) for some \( t > 0 \).
Then $\delta (0) \neq \gamma (0)$. In particular:

$$d (\gamma (-1), S (1)) < 2.$$ 

Otherwise, the concatenation of $[-1,0]$ and $[0,1]$ would be a distance length minimizing path hence a geodesic.

Now pick $N$ with

$$d (\gamma (t_N), S (t_N)) < 2 - d (\gamma (-1), S (1)).$$

And consider the path:

$$\gamma (t_N) \overset{S (t_N)}{\longrightarrow} \gamma (t_N)$$

consists of a length min. geodesic from $\gamma (-1)$ to $S (1)$, the segment $S [0, t_N]$ and a length min. geodesic from $S (t_N)$ to $\gamma (t_N)$. 

\[\text{Diagram:}\]

A line segment from $\gamma (-1)$ to $S (1)$, followed by a line segment from $S (t_N)$ to $\gamma (t_N)$.
This path has length \( t_n + 1 = d(x_{n+1}, x_{n+2}) \) which is a contradiction. Thus \( f(1) = n + 1 \forall t \)

That is \( \exp(2t \text{ ad}(x_i)(x)) = \exp(2t \cdot x), \forall t \)

hence \( \text{ad}(x_i)(x) = x, \forall x \in A \).

But this implies \( \dim \mathfrak{z}(g) \geq 1 \) contradicting semi-simplicity of \( g \). \( \square \)