Now we turn to a global property of riemannian symmetric spaces of non-compact type.

<u>Thm IV. 13.</u> Let (G, K) be an effective riemannian symmetric pair of non-compact type and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition. Then

(1) K is connected and $Z(G) \subset K$. Moreover K is compact if and only if $Z(G)$ is finite.

(2) Let $M = G/K$, $o = eK$ and $\mathrm{Exp}_o : T_o M \to M$ the riemannian exponential map. Then Exp_o is a diffeomorphism.

(3) The mapping
$$\mathfrak{p} \times K \longrightarrow G$$
$$(X, k) \longmapsto (\exp X) . k$$
is a diffeomorphism.

We will need the following important construction, often used in the sequel.

Define:

$$\langle Z_1, Z_2 \rangle := - B_{\mathfrak{g}}\left(Z_1, \theta(Z_2) \right).$$

Then, writing $Z_i = X_i + Y_i$ with $X_i \in \mathfrak{k}$, $Y_i \in \mathfrak{p}$ we have:

$$\langle Z_1, Z_2 \rangle = - B_{\mathfrak{g}}\left(X_1 + Y_1, X_2 - Y_2 \right)$$

$$= - B_{\mathfrak{g}}(X_1, X_2) + B_{\mathfrak{g}}(Y_1, Y_2)$$

which implies, since $B_{\mathfrak{g}}|_{\mathfrak{k} \times \mathfrak{k}} \ll 0$ and $B_{\mathfrak{g}}|_{\mathfrak{p} \times \mathfrak{p}} \gg 0$ that $\langle \ \rangle$ is a scalar product on $\mathfrak{g}$.

Lemma IV. 14.

(1) $\forall\, X \in \mathfrak{p}$, $\operatorname{ad} X$ is symmetric.

(2) $\forall\, T \in \mathfrak{k}$, $\operatorname{ad} T$ is skew-symmetric

(3) $\forall\, k \in K$, $\operatorname{Ad}(k)$ is orthogonal.

Proof: (1) Let $X \in \mathfrak{p}$, $Y, Z \in \mathfrak{g}$:

$$\begin{aligned}
\langle \operatorname{ad}(X)\, Y, Z \rangle &= - B_{\mathfrak{g}}([X,Y], \theta(Z)) \\
&= B_{\mathfrak{g}}(Y, [X, \theta(Z)]) \\
&= - B_{\mathfrak{g}}(Y, \theta([X,Z])) \\
&= \langle Y, \operatorname{ad}(X)\, Z \rangle .
\end{aligned}$$

The proofs of (2) and (3) are analogous.

$\boxed{\xi}$

Proof of Thm IV. 13

Recall that $(G^\sigma)^\circ \subset K \subset G^\sigma$ where σ is as in the definition of a RSP; it follows that (G, K°) is an RSP as well.

Let $M = G/K^\circ$ with a G-invariant riemannian metric, $o = eK^\circ \in M$ the basepoint, $\pi : G \to M$, $g \mapsto g . o$ and

$$\mathrm{Exp} : \mathfrak{p} \xrightarrow[D_e\pi]{} T_oM \xrightarrow[\mathrm{Exp}_o]{} M$$

where Exp_o is the riemannian exponential.

Then: $\mathrm{Exp}\, X = (\exp X)_* o$.

Thus since M is complete, Exp_o is surjective hence $\forall g \in G\ \exists X \in \mathfrak{p}$ with

$$g o = (\exp X)_* o$$

and hence $\exists k \in K^\circ$, $g = (\exp X) \cdot k$.

If we now define $\varphi : \mathfrak{p} \times K \longrightarrow G$

$$(X, k) \longmapsto (\exp X) \cdot k$$

then we have just shown that

$$\varphi(\mathfrak{p} \times K^{\circ}) = G.$$

We proceed now to show that the map $\varphi : \mathfrak{p} \times K \longrightarrow G$ is injective.

Let $X_1, X_2 \in \mathfrak{p}$ and $k_1, k_2 \in K$ with $(\exp X_1) k_1 = (\exp X_2) k_2$.

Applying Ad we get:

$$e^{ad(X_1)} Ad(k_1) = e^{ad(X_2)} Ad(k_2).$$

If T^* denotes the adjoint on $T \in \operatorname{End} \mathfrak{g}$ wrt the scalar product, it follows that ~~from~~

$$Ad(k_1)^* e^{ad(X_1)} = Ad(k_2)^* e^{ad(X_2)}$$

and hence by multiplying both equalities

$$e^{2\,\mathrm{ad}(X_1)} = e^{2\,\mathrm{ad}(X_2)}. \qquad (*)$$

Now for $X \in \mathfrak{p}$, $\mathrm{ad}(X)$ is symmetric and $e^{\mathrm{ad}(X)}$ is symmetric positive definite; it has hence a unique positive definite n'th root $\forall n \in \mathbb{N}^*$, which is $e^{\frac{1}{n}\mathrm{ad}(X)}$. Thus we deduce from $(*)$ that $e^{r\,\mathrm{ad}(X_1)} = e^{r\,\mathrm{ad}(X_2)}$ $\forall r \in \mathbb{Q}$ and hence $\mathrm{ad}(X_1) = \mathrm{ad}(X_2)$ which implies $X_1 = X_2$ since $\mathfrak{g}$ is semisimple.

Thus $k_1 = k_2$ and φ is injective.

This implies $K = K^0$ and φ is bijective.

Now we show that $Z(G) \subset K$.

Let $z \in Z(G)$ and write $z = (\exp X) \cdot k$

with $X \in \mathfrak{p}$, $k \in K$. Then:

$$Id = Ad(k) = e^{ad(X)} Ad(k)$$

which by the argument above implies

$$e^{2ad(X)} = Id$$

hence $e^{r\,ad(X)} = Id$ $\forall r \in \mathbb{Q}$ and

hence $ad(X) = 0$, implying $X = 0$.

Thus $z = k \in K$.

Thus we obtain an exact sequence of topological groups

$$(e) \to Z(G) \to K \to Ad(K) \to \{Id\}.$$

We know that $Ad(K)$ is compact; moreover since (G, K) is effective $Z(G) = Z(G) \cap K$ is discrete. Thus K is compact if and only if $Z(G)$ is finite.

Thus we have shown that φ is bijective which implies that $\mathrm{Exp}_0 : T_0 M \to M$ is bijective. In order to show that the latter is regular at every point we show that $\mathrm{Exp} : \mathfrak{p} \longrightarrow M$ has this property and to this end use the formula (see Cor. II.41) for the derivative $D_X \mathrm{Exp}$ established in 2.5:

$$D_X \mathrm{Exp} = \left(D_0 L_{\exp X} \circ D_e \pi \right) \left(\sum_{n=0}^{\infty} \frac{\left(T_X |_{\mathfrak{p}} \right)^n}{(2n+1)!} \right)$$

where $T_X = (\mathrm{ad}\, X)^2$, $X \in \mathfrak{p}$.

The following two observations are crucial:

(1) For X, Y, Z in $\mathfrak{p}$:

$$B_g(T_X Y, Z) = B_g(Y, T_X Z)$$

(2) $\forall\, X, Y \in \mathfrak{p}$:

$$B_{\mathfrak{g}}(T_X Y, Y) = B_{\mathfrak{g}}((\operatorname{ad} X)^2 Y, Y)$$

$$= -B_{\mathfrak{g}}(\operatorname{ad}(X)(Y), \operatorname{ad}(X)(Y))$$

$$= -B_{\mathfrak{g}}([X,Y],[X,Y])$$

$$\geq 0$$

since $[X,Y] \in \mathfrak{k}$ and $B_{\mathfrak{g}}\big|_{\mathfrak{k}\times\mathfrak{k}} << 0$.

Thus $T_X|_{\mathfrak{p}}$ and hence $(T_X|_{\mathfrak{p}})^n$ $\forall n \geq 1$ is symmetric semipositive definite which implies

$$\det\left(\sum_{n=0}^{\infty} \frac{(T_X|_{\mathfrak{p}})^n}{(2n+1)!}\right) \geq 1$$

hence does not vanish.

The regularity of φ is left as an exercise. $\square$

Remark IV. 14.

(1) To show the regularity of Exp_o we could have used Thm IV. 2 (2) which says that since (G,K) is of non-compact type the sectional curvature of $M = G/K$ is everywhere ≤ 0 which by Hadamard's theorem IV. 1 implies that $\mathrm{Exp}_o : T_o M \to M$ is a covering. The argument presented above is more direct and it is used for establishing that φ is regular.

(2) It follows from Hadamard's theorem and Thm IV. 13 that for a symmetric space M of non-compact type,
$\mathrm{Exp}_o : T_o M \to M$ is a distance increasing diffeomorphism.

IV.3 From OSL's to RSP's to RSS's

In this subsection we want to apply the techniques developed so far to study how we can reverse the process that goes from Riemannian symmetric spaces to orthogonal symmetric Lie algebras via riemannian symmetric pairs. Our first step is to go from reduced semisimple OSL's to reduced semisimple RSP's.

Theorem IV.18 Let $(\mathfrak{g}, \Theta)$ be a reduced semisimple OSL. Let:

$$G = \mathrm{Aut}(\mathfrak{g})^0 ; \quad \sigma(\alpha) := \Theta \alpha \Theta^{-1}, \alpha \in G$$

and $(G^\sigma)^0 < K < G^\sigma$. Then (G, K) is a RSP whose associated OSL is isomorphic to $(\mathfrak{g}, \theta)$. Moreover G^σ is compact, $Z(G) = e$ and G acts faithfully on G/K.

<u>Proof</u>: First we use Thm III.22 and write $(\mathfrak{g}, \theta) \simeq \prod_{i=1}^{a} (\mathfrak{g}_i, \theta_i)$ as a product of irreducible components. It is left as an exercise that under this isomorphism:

$$(\mathrm{Aut}\,\mathfrak{g})^\circ \xrightarrow{\sim} \prod_{i=1}^{a} (\mathrm{Aut}\,\mathfrak{g}_i)^\circ$$

$$G^\sigma \xrightarrow{\sim} \prod_{i=1}^{a} G^{\sigma_i}$$

and hence we may assume that $(\mathfrak{g}, \theta)$ is irreducible.

Recall that $\theta \in \mathrm{Aut}\,\mathfrak{g}$ is an involution;

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition.

We define a scalar product on $\mathfrak{g}$ as follows:

(1) If $\mathfrak{g}$ is of compact type

$$\langle \, , \rangle = - B_{\mathfrak{g}}$$

(2) If $\mathfrak{g}$ is of non-compact type

$$\langle Z_1, Z_2 \rangle = - B_{\mathfrak{g}}\big(Z_1, \theta(Z_2)\big) \qquad Z_i \in \mathfrak{g}$$

(compare with the disc. preceeding lemma IV.14).

Now since $\mathfrak{g}$ is semisimple it follows from Prop. IV.9 that

$$\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathrm{Der}\, \mathfrak{g} = \mathrm{Lie}\, G$$

is an isomorphism.

Next $G^{\sigma} = \{ \alpha \in G : \alpha\theta = \theta\alpha \}$

is clearly a closed subgroup of G.

We claim that G^σ is compact. Indeed G^σ preserves the scalar product $\langle \, , \rangle$. This is clear if $\mathfrak{g}$ is of compact type since $\langle \; \rangle = -B_{\mathfrak{g}}$ and $\mathrm{Aut}(\mathfrak{g})$ preserves $B_{\mathfrak{g}}$ (lemma IV.2). If $\mathfrak{g}$ is of non-compact type we have $\forall \alpha \in G^\sigma$:

$$\begin{aligned}\langle \alpha Z_1, \alpha Z_2 \rangle &= -B_{\mathfrak{g}}(\alpha Z_1, \theta \alpha Z_2) \\ &= -B_{\mathfrak{g}}(\alpha Z_1, \alpha \theta Z_2) \\ &= -B_{\mathfrak{g}}(Z_1, \theta Z_2) = \langle Z_1, Z_2 \rangle.\end{aligned}$$

Let's compute the Lie algebra of G^σ:

$$\mathrm{Lie}\, G^\sigma = \{ D \in \mathrm{Der}\, \mathfrak{g} : D\theta = \theta D \}$$

$$\overset{\text{Prop. IV.9}}{=} \{ \mathrm{ad}_{\mathfrak{g}}(X) : \theta \, \mathrm{ad}_{\mathfrak{g}}(X) = \mathrm{ad}_{\mathfrak{g}}(X)\theta \}$$

$$= \{ \mathrm{ad}_{\mathfrak{g}}(X) : \mathrm{ad}_{\mathfrak{g}}(\theta(X)) = \mathrm{ad}_{\mathfrak{g}}(X) \}$$

$$= \mathrm{ad}_{\mathfrak{g}}\, \mathfrak{k}.$$

Hence $\mathrm{Lie}\, K = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$. Thus $\mathrm{ad}_{\mathfrak{g}}$ establishes an isomorphism between $(\mathfrak{g}, \theta)$ and the OSL associated to (G, K).

Now to the last assertion: We know that (G, K) is reduced, hence the kernel N of the G-action on G/K is discrete. Since G is connected and N discrete normal, we have $N \subset Z(G)$. Finally let $\alpha \in Z(G)$, thus: $\alpha\beta = \beta\alpha$ $\forall \beta \in \mathrm{Aut}(\mathfrak{g})^{\circ}$; passing to the Lie algebra this implies

$$\alpha\, \mathrm{ad}_{\mathfrak{g}}(X) = \mathrm{ad}_{\mathfrak{g}}(X)\, \alpha \quad \forall X \in \mathfrak{g}$$

or

$$\mathrm{ad}_{\mathfrak{g}}(\alpha(X)) = \mathrm{ad}_{\mathfrak{g}}(X) \quad \forall X \in \mathfrak{g}$$

hence $\alpha = \mathrm{id}$.

<u>Remark IV. 16</u> : One could avoid using the decomposition theorem by using a theorem of Whitney saying that a real algebraic set has finitely many connected components. Observe that

$$\mathrm{Aut}\,\mathfrak{g} = \{ \alpha \in GL(\mathfrak{g}) : \alpha([X_i, X_j]) = [\alpha(X_i), \alpha(X_j)] \}$$

where $X_1, \dots, X_n$ is a basis of $\mathfrak{g}$ and hence is defined by finitely many quadratic equations in the matrix entries of α. Thus $(\mathrm{Aut}\,\mathfrak{g})^\circ = G$ has finite index in $\mathrm{Aut}\,\mathfrak{g}$. Now $G^\sigma = G \cap (\mathrm{Aut}\,\mathfrak{g})^\sigma$ is an open subgroup of $(\mathrm{Aut}\,\mathfrak{g})^\sigma$, hence $(G^\sigma)^\circ = ((\mathrm{Aut}\,\mathfrak{g})^\sigma)^\circ$. Now again, $(\mathrm{Aut}\,\mathfrak{g})^\sigma$ is real algebraic and hence

$(G^\sigma)^0$ has finite index in $(\mathrm{Aut}\,\mathfrak{g})^\sigma$ and hence in G^σ, since $G^\sigma \subset (\mathrm{Aut}\,\mathfrak{g})^\sigma$.
Now we go back in the proof of the theorem to the point where we show:

$$\mathrm{Lie}\, G^\sigma = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$$

which implies since $\mathfrak{k}$ is compactly embedded that $(G^\sigma)^0$ is compact. Together with the fact that $(G^\sigma)^0$ has finite index in G^σ this implies that G^σ is compact.

Now we can partially answer a natural question, namely how many "different" riemannian symmetric spaces are there with the same OSL? We have seen that in the case of compact type there may be several, as for example S^n and $P^n(\mathbb{R})$ lead to the same OSL. However:

<u>Corollary IV. 17</u>. Let M_1, M_2 be RSS of noncompact type with associated OSL's $(\mathfrak{g}_1, \theta_1)$, $(\mathfrak{g}_2, \theta_2)$. Assume that

(1) $(\mathfrak{g}_1, \theta_1) \cong (\mathfrak{g}_2, \theta_2)$

(2) M_i is endowed with the riemannian metric coming from $B_{\mathfrak{g}_i}$.

Then M_1 is isometric to M_2.

We leave this as an interesting exercise that uses Thm IV. 13 and Thm IV. 15.

Given now a RSP (G, K), where G acts moreover effectively on $M := G/K$, we have thus $G \subset Is(M)^0$: when is there equality? Clearly this is not always the case as the example of $G = \mathbb{R}^n$, $\sigma(v) = -v$, shows, since $M = \mathbb{E}^n$ and $Is(M)^0 = \mathbb{R}^n \rtimes SO(n)$.

But these are essentially the only examples:

Thm IV. 18

Let (G, K) be a RSP and assume that G is semisimple and acts ~~effectively~~ faithfully on $M = G/K$. Then $G = Is(M)^0$ and

$$K = Stab_G(o), \quad o = eK.$$

<u>Proof:</u> Let $G_0 := Is(M)^0$ and

$$\tau : G \to G_0$$

given by $\tau(g)xK = gxK$. Then τ is a smooth injective homomorphism.

Let $\sigma : G \circlearrowleft$ be the involution such that $(G^\sigma)^0 \subset K \subset G^\sigma$, $\pi : G \to M$, $g \mapsto g.o$ where $o = eK$ and $s_o : M \to M$ the geodesic symmetry at o. Recall from Thm II.28 that $s_o \pi = \pi \cdot \sigma$.

Define $\sigma_0 : G_0 \to G_0$ by $\sigma_0(\alpha) = s_o \alpha s_o^{-1}$ and $K_0 := \underset{G_0}{Stab}(o)$; then we know (Prop. II.25) that (G_0, K_0) is an RSP and it follows from $s_o \pi = \pi \sigma$ that the diagram:

$$\begin{array}{ccc} G & \xrightarrow{\sigma} & G \\ \tau \downarrow & & \downarrow \tau \\ G_0 & \xrightarrow[\sigma_0]{} & G_0 \end{array} \qquad (*)$$

commutes.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the OSL $(\mathfrak{g}, D_e\sigma)$ and

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

the one of the OSL $(\mathfrak{g}_0, D_e\sigma_0)$.

Then it follows from (*) that

$$D_e\tau(\mathfrak{p}) \subset \mathfrak{p}_0, \quad D_e\tau(\mathfrak{k}) \subset \mathfrak{k}_0.$$

Since $M = G/K = G_0/K_0$ we have

$\dim \mathfrak{p} = \dim \mathfrak{p}_0$ and hence $D_e\tau(\mathfrak{p}) = \mathfrak{p}_0$

since $D_e\tau$ is injective.

Now we make an important observation valid for all reduced semisimple OSL's namely

$$[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}.$$

Indeed: one verifies that $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ is an ideal in $\mathfrak{g}$; its orthogonal for $B_\mathfrak{g}$ is contained in the orthogonal of $\mathfrak{p}$ which is $\mathfrak{h}_3$. Hence it vanishes which implies $\mathfrak{p} + [\mathfrak{p},\mathfrak{p}] = \mathfrak{g}$ which with $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{h}_3$ implies $[\mathfrak{p},\mathfrak{p}] = \mathfrak{h}_3$.

Thus we deduce from $[\mathfrak{p},\mathfrak{p}] = \mathfrak{h}_3$ and $D_e\tau(\mathfrak{p}) = \mathfrak{p}_0$ that:

$$D_e\tau(\mathfrak{h}_3) = [\mathfrak{p}_0,\mathfrak{p}_0].$$

Next, we observe that $(\mathfrak{g}_0, D_e\sigma_0)$ is a reduced hence effective OSL and lemma III.10 implies then that the null space $\mathfrak{e}_0$ of the Killing form $B_{\mathfrak{g}_0}$ is contained in $\mathfrak{p}_0$ and is an abelian ideal in $\mathfrak{g}_0$. We have the inclusion

$$\mathfrak{e}_0 \subset \mathfrak{p}_0 = D_e\tau(\mathfrak{p}) \subset D_e\tau(\mathfrak{g}) \subset \mathfrak{g}'$$

and hence $\mathfrak{e}_0$ is an abelian ideal in $D_e\tau(\mathfrak{g})$ as well. Since $D_e\tau(\mathfrak{g}) \cong \mathfrak{g}$

is semisimple this implies $e_0 = 0$. Hence $(\mathfrak{g}_0, D_e\sigma_0)$ is a reduced semisimple OSL and hence $[\mathfrak{p}_0, \mathfrak{p}_0] = \mathfrak{k}_0$ which implies $D_e\tau(\mathfrak{k}) = \mathfrak{k}_0$. Thus τ induces an isomorphism between the OSL's

$$(\mathfrak{g}, D_e\sigma) \xrightarrow{\sim} (\mathfrak{g}_0, D_e\sigma_0)$$

and hence τ is a Lie group isomorphism.

$\boxed{}$