IV. 4. Duality.

There is a remarkable and important duality between compact and non compact type OSL's that is a special case of a general construction we outline here. First we need some preliminaries on complexification of real vector spaces and real Lie algebras. A complex structure on a real vector space V is given by an endomorphism $J \in \mathrm{End}\, V$ with $J^2 = -\mathrm{Id}$. The formula:

$$(x+iy).v := xv + yJv$$

for $x, y \in \mathbb{R}$, $v \in V$ ~~gives~~ defines a $\mathbb{C}$-vector space structure on V denoted $\widetilde{V}$. Conversely a $\mathbb{C}$-vector space E can be considered as $\mathbb{R}$-vector space, denoted $E^{\mathbb{R}}$, with complex structure $Jv := iv$ and evidently $\widetilde{(E^{\mathbb{R}})} = E$.

An $\mathbb{R}$-Lie algebra $\mathfrak{v}$ has a $\mathbb{C}$-structure if $\mathfrak{v}$ as $\mathbb{R}$-vector space has a $\mathbb{C}$-structure $J \in \operatorname{End} V$ and the Lie bracket verifies

$$[X, JY] = J[X, Y] \quad \forall X, Y \in \mathfrak{v}.$$

In this case $[\ ,\ ]$ is bilinear for the $\mathbb{C}$-vector space structure $\tilde{\mathfrak{v}}$ and $\tilde{\mathfrak{v}}$ is thus a $\mathbb{C}$-Lie algebra.

Now given an $\mathbb{R}$-vector space W define on $W \times W$, $J(v, w) := (-w, v)$. Then $J \in \operatorname{End}(W \times W)$ and $J^2 = -\mathrm{Id}$. We denote the $\mathbb{C}$-vector space $\widetilde{W \times W}$ by $W^{\mathbb{C}}$ and call it the complexification of W.

The complex conjugation on $W^{\mathbb{C}}$ is the $\mathbb{R}$-linear endomorphism $\tau(v, w) = (v, -w)$.

It is convenient to identify W with an $\mathbb{R}$-subspace of $W^{\mathbb{C}}$ by:

$$\begin{aligned} W &\longrightarrow W^{\mathbb{C}} \\ v &\longmapsto (v, 0). \end{aligned}$$

Then $(W^{\mathbb{C}})^{\mathbb{R}} = W + iW$. Thus every $Z \in W^{\mathbb{C}}$ is uniquely representable as

$$Z = X + iY \qquad X, Y \in W$$ and

$$\tau(Z) = X - iY.$$

If ℓ_0 is a $\mathbb{R}$-Lie algebra, then (exercise) the Lie bracket $[\ \ ]$ extends uniquely to a $\mathbb{C}$-bilinear Lie bracket

$$\ell \times \ell \longrightarrow \ell$$

where $\ell = \ell_0^{\mathbb{C}}$.

Let $T \in \operatorname{End} V$; we define

$$T^{\mathbb{C}}(v + iw) = Tv + iTw$$

Then $T^{\mathbb{C}} \in \operatorname{End} V^{\mathbb{C}}$ is an endomorphism of the $\mathbb{C}$ – vector space $V^{\mathbb{C}}$. Clearly :

$$(1) \quad (\lambda_1 T_1 + \lambda_2 T_2)^{\mathbb{C}} = \lambda_1 T_1^{\mathbb{C}} + \lambda_2 T_2^{\mathbb{C}}$$

$$T_i \in \operatorname{End} V, \quad \lambda_i \in \mathbb{R}.$$

$$(2) \quad (T_1 T_2)^{\mathbb{C}} = T_1^{\mathbb{C}} T_2^{\mathbb{C}}.$$

Then one verifies $\operatorname{tr}_V(T) = \operatorname{tr}_{V^{\mathbb{C}}}(T^{\mathbb{C}})$.

(IV, 19)

We also need that if $A \in \operatorname{End}(V^{\mathbb{C}})$ then $\operatorname{tr}_{(V^{\mathbb{C}})^{\mathbb{R}}} A = 2 \operatorname{Re} \operatorname{tr}_{V^{\mathbb{C}}} A$. (IV. 20).

Indeed if $v_1, \ldots, v_n$ is a $\mathbb{C}$ – basis of $V^{\mathbb{C}}$ then $v_1, \ldots, v_n, iv_1, \ldots, iv_n$ is a $\mathbb{R}$ – basis and if $a \in M_n(\mathbb{C})$

is the matrix of A in $v_1, \dots, v_n$ then writing $a = b + ic$ $b, c \in M_n(\mathbb{R})$, the matrix of A in the basis $v_1, \dots, v_n$, $iv_1, \dots, iv_n$ is:

$$\begin{pmatrix} b & -c \\ c & b \end{pmatrix}$$

from which the relation follows.

Let now $\mathfrak{g}_0$ be an $\mathbb{R}$-Lie algebra $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$ its complexification and

$$e : \mathfrak{gl}(\mathfrak{g}_0) \longrightarrow \mathfrak{gl}(\mathfrak{g})$$
$$T \longmapsto T^{\mathbb{C}}.$$

Then e is a homomorphism of real Lie algebras and the diagramm:

$$\begin{array}{ccc} \mathfrak{g}_0 & \xrightarrow{ad_{\mathfrak{g}_0}} & \mathfrak{gl}(\mathfrak{g}_0) \\ \downarrow & & \downarrow e \\ \mathfrak{g} & \xrightarrow[ad_{\mathfrak{g}}]{} & \mathfrak{gl}(\mathfrak{g}) \end{array}$$

commutes.

Indeed: let $X \in \mathfrak{g}_0$, $T = Y + iZ \in \mathfrak{g}$.

Then:

$$\mathrm{ad}_{\mathfrak{g}}(X)\,T = [X, Y + iZ]$$

$$= [X, Y] + i\,[X, Z]$$

$$= e(\mathrm{ad}_{\mathfrak{g}_0}(X))(T).$$

With these preliminaries one can easily show

<u>Lemma IV.21</u>

(1) $\forall X, Y \in \mathfrak{g}_0$, $B_{\mathfrak{g}_0}(X, Y) = B_{\mathfrak{g}}(X, Y)$

(2) $\forall X, Y \in \mathfrak{g}$: $B_{\mathfrak{g}^{\mathbb{R}}}(X, Y) = 2\,\mathrm{Re}\,B_{\mathfrak{g}}(X, Y)$

(3) $\mathfrak{g}_0$ is semisimple $\Leftrightarrow$ $\mathfrak{g}$ is semisimple

$\Leftrightarrow$ $\mathfrak{g}^{\mathbb{R}}$ is semisimple.

Let now $(\mathfrak{g}_0, \Theta_0)$ be an OSL,

$\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$ and $\Theta = \Theta_0^{\mathbb{C}}$ as well as

τ the complex conjugation on $\mathfrak{g}$.

Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition of $\mathfrak{g}_0$. Then we have the following $\mathbb{R}$-subspaces of $\mathfrak{g}^{\mathbb{R}}$:

$$\mathfrak{k},\ i\mathfrak{k},\ \mathfrak{p},\ i\mathfrak{p}$$

with the following ~~comm~~ bracket relations:

$$[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

$$[\mathfrak{k}, i\mathfrak{p}] \subset i\mathfrak{p} \qquad [i\mathfrak{p}, i\mathfrak{p}] = [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

As a result

$$\mathfrak{g}^* := \mathfrak{k} + i\mathfrak{p}$$

is a Lie subalgebra of $\mathfrak{g}^{\mathbb{R}}$. It is moreover τ invariant; define:

$$\Theta^* := \tau|_{\mathfrak{g}^*}.$$

We have

Proposition IV.22

(1) $(\mathfrak{g}^*, \theta^*)$ is an OSL.

(2) $(\mathfrak{g}^*)^{\mathbb{C}} \simeq \mathfrak{g}$ and $(\theta^*)^{\mathbb{C}} = \theta^{\mathbb{C}}$

(3) $(\mathfrak{g}^*, \theta^*)$ is effective $\Leftrightarrow$ $(\mathfrak{g}_0, \theta_0)$ is effective.

(4) $(\mathfrak{g}^*, \theta^*)$ is reduced $\Leftrightarrow$ $(\mathfrak{g}_0, \theta_0)$ is reduced.

(5) $(\mathfrak{g}^*, \theta^*)$ is of compact type

$\Longleftrightarrow$ $(\mathfrak{g}_0, \theta_0)$ is of non-compact type.

Proof:

(1) Clearly since $\tau \in \mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$,

$\theta^* \in \mathrm{Aut}(\mathfrak{g}^*)$ with fixed point algebra $\mathfrak{h}$. We have to check that $\mathfrak{h}$ is compactly embedded in $\mathfrak{g}^*$.

We consider the $\mathbb{R}$-vector space isomorphism

$$\varphi : \mathfrak{g}_0 \to \mathfrak{g}^*$$
$$X + Y \mapsto X + iY$$

Then: $\varphi \circ \mathrm{ad}_{\mathfrak{g}_0} Z = \mathrm{ad}_{\mathfrak{g}^*} Z \circ \varphi \quad \forall Z \in \mathfrak{z}$.

Since $\mathfrak{z} < \mathfrak{g}_0$ is compactly embedded, there is $U < GL(\mathfrak{g}_0)$ compact connected with $\mathrm{ad}_{\mathfrak{g}_0} \mathfrak{z} = \mathrm{Lie}\, U$. Now consider

$$\Phi : GL(\mathfrak{g}_0) \longrightarrow GL(\mathfrak{g}^*)$$
$$A \longmapsto \varphi A \varphi^{-1}$$

Then Φ is a Lie group isomorphism whose derivative is

$$(D_I \Phi)(B) = \varphi B \varphi^{-1};$$

and $D_I \Phi (\mathrm{ad}_{\mathfrak{g}_0} \mathfrak{z}) = \mathrm{ad}_{\mathfrak{g}^*} \mathfrak{z}$.

Thus $\mathrm{ad}_{\mathfrak{g}^*} \mathfrak{z} = \mathrm{Lie}\, \Phi(U)$ and $\Phi(U) \leq GL(\mathfrak{g}^*)$ is compact connected.

(2) Verification left to the reader.

(3) Follows from $\mathcal{Z}(\mathfrak{g}_0) \cap \mathfrak{k} = \mathcal{Z}(\mathfrak{g}^*) \cap \mathfrak{k}$.

(4) An ideal $\mathfrak{n}$ of $\mathfrak{g}_0$ is contained in $\mathfrak{k}$ iff $\mathfrak{n}$ is an ideal in $\mathfrak{k}$ and $[\mathfrak{n}, \mathfrak{p}] = 0$.
Thus if $\mathfrak{n} \triangleleft \mathfrak{k}$ with $[\mathfrak{n}, i\mathfrak{p}] = 0$, we have $\mathfrak{n} \triangleleft \mathfrak{k}$ with $[\mathfrak{n}, \mathfrak{p}] = 0$ which shows that if $(\mathfrak{g}_0, \theta_0)$ is reducible so is $(\mathfrak{g}, \theta)$.

(5) Since $(\mathfrak{g}_0, \theta_0)$ is effective semisimple so is $(\mathfrak{g}^*, \theta^*)$ by lemma IV.21 and (3) above. We have by lemma IV.21:

$\forall X, Y \in \mathfrak{p}$:

$$B_{\mathfrak{g}_0}(X, Y) = B_{\mathfrak{g}}(X, Y) = -B_{\mathfrak{g}}(iX, iY)$$
$$= -B_{\mathfrak{g}^*}(iX, iY)$$

since $(\mathfrak{g}^*)^{\mathbb{C}} \simeq \mathfrak{g}$. $\boxed{}$