Def. IV.23 The OSL $(\mathfrak{g}^*, \theta^*)$ given by Prop. IV.22 is the dual OSL of $(\mathfrak{g}_0, \theta_0)$.

Example IV.24

Let's determine the dual of $(\mathfrak{sl}(n,\mathbb{R}), \theta)$ where $\theta(X) = -{}^tX$.

We identify $\mathfrak{sl}(n,\mathbb{R})^{\mathbb{C}}$ with $\mathfrak{sl}(n,\mathbb{C})$.

Now $\mathfrak{k} = \{ X \in \mathfrak{sl}(n,\mathbb{R}) : X + {}^tX = 0 \}$

$\mathfrak{p} = \{ X \in \mathfrak{sl}(n,\mathbb{R}) : X = {}^tX \}$.

Then

$$\mathfrak{k} + i\mathfrak{p} = \{ Z \in \mathfrak{sl}(n,\mathbb{C}) : Z = X + iY, \; {}^tX + X = 0, \; {}^tY = Y \}$$

$$= \{ Z \in \mathfrak{sl}(n,\mathbb{C}) : Z + {}^t\overline{Z} = 0 \}$$

$$= \mathfrak{su}(n).$$

Let $M = {SL(n,\mathbb{R})}/{SO(n)}$ and $M^* = {SU(n)}/{SO(n)}$

then the OSL associated to M is

$(sl(n,\mathbb{R}), \theta)$ while the OSL associated

to M^* is $(su(n), \theta^*)$, its dual.

<u>Example IV. 25</u>

Let $g_0 = so(n) = \{ X \in gl(n,\mathbb{R}) : X + {}^tX = 0 \}$

and p, q integers ≥ 1 with $n = p+q$.

Let $\Theta_0(X) = I_{p,q} \, X \, I_{p,q}$ where

$$I_{p,q} = \begin{pmatrix} -Id_p & 0 \\ 0 & Id_q \end{pmatrix}.$$

Then one verifies easily that $\Theta_0(so(n)) = so(n)$

and $\Theta_0^2 = Id$.

We can write the matrices in $so(n)$ in $p \times q$ bloc form:

$$so(n) = \left\{ X = \begin{pmatrix} A & B \\ -{}^tB & D \end{pmatrix} : \begin{array}{l} A + {}^tA = 0 \\ D + {}^tD = 0 \end{array} \right\}$$

Then for $X = \begin{pmatrix} A & B \\ -{}^tB & D \end{pmatrix}$,

$$\theta_0(X) = \begin{pmatrix} A & -B \\ {}^tB & D \end{pmatrix}.$$

Thus
$$\mathfrak{k}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in so(p),\ D \in so(q) \right\}$$

$$\mathfrak{p}_0 = \left\{ \begin{pmatrix} 0 & B \\ -{}^tB & 0 \end{pmatrix} : B \in M_{p,q}(\mathbb{R}) \right\}.$$

Now $\mathfrak{g}^* = \mathfrak{k}_0 + i\,\mathfrak{p}_0 \subset gl(n, \mathbb{C})$.

$$= \left\{ \begin{pmatrix} X_1 , & i X_2 \\ -i\,{}^tX_2 , & X_3 \end{pmatrix} : \begin{array}{l} X_1 \in so(p) \\ X_3 \in so(q) \\ X_2 \in M_{p,q}(\mathbb{R}) \end{array} \right\}$$

Let $\sigma : gl(n, \mathbb{C}) \circlearrowleft$ be defined by

$$\sigma(Y) = \begin{pmatrix} -iI_p & 0 \\ 0 & I_q \end{pmatrix} Y \begin{pmatrix} iI_p & 0 \\ 0 & I_q \end{pmatrix}$$

Then

$$\sigma \begin{pmatrix} X_1 & iX_2 \\ -i\,{}^tX_2 & X_3 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ {}^tX_2 & X_3 \end{pmatrix}$$

and $\sigma : g^* \longrightarrow so(p,q)$ is an isomorphism with the real Lie algebra $so(p,q)$. The Cartan involution corresponding to Θ^* is then

$$\begin{pmatrix} X_1 & X_2 \\ {}^tX_2 & X_3 \end{pmatrix} \to \begin{pmatrix} X_1 & -X_2 \\ -{}^tX_2 & X_3 \end{pmatrix}.$$

Thus $M = SO(n) / SO(p) \times SO(q)$ is a symmetric space with OSL $(so(n), \Theta_0)$

And $M^* = SO(p,q) / SO(p) \times SO(q)$ is a symm. space with associated OSL (g^*, Θ^*).

We want to show now how this duality can be realized at the level of riemannian symmetric pairs. To this end we will use the construction in Thm IV. 15.

Let $(\mathfrak{g}_0, \theta_0)$ be a reduced semisimple OSL; let $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$ and τ the complex conjugation wrt $\mathfrak{g}_0$. Recall that $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p}$ and $\theta^* = \tau|_{\mathfrak{g}^*}$. Then $\mathfrak{g}^{*\mathbb{C}} = \mathfrak{g}$ and $\theta^{*\mathbb{C}} = \theta^{\mathbb{C}}$.

Consider
$$e_0 : \operatorname{Aut} \mathfrak{g}_0 \longrightarrow \operatorname{Aut} \mathfrak{g}$$
$$\alpha \longmapsto \alpha^{\mathbb{C}}$$
which is an injective Lie group morphism evidently satisfying:
$$e_0(\theta_0 \alpha \theta_0^{-1}) = \theta_0^{\mathbb{C}} \alpha^{\mathbb{C}} (\theta_0^{\mathbb{C}})^{-1}.$$

Call σ_0 the restriction to $e_0(\mathrm{Aut}\,\mathfrak{g}_0)$ of the conjugation by $\theta_0^{\mathbb{C}}$.

Analogously consider

$$e^* : \mathrm{Aut}\,\mathfrak{g}^* \longrightarrow \mathrm{Aut}\,\mathfrak{g}$$
$$\beta \longmapsto \beta^{\mathbb{C}}$$

which satisfies

$$e^*(\theta^* \alpha\, \theta^{*-1}) = (\theta^*)^{\mathbb{C}} \alpha ((\theta^*)^{\mathbb{C}})^{-1}$$

and call σ^* the restriction to $e^*(\mathrm{Aut}\,\mathfrak{g}^*)$ of the conjugation by $(\theta^*)^{\mathbb{C}}$.

<u>Proposition IV.26:</u>

The groups $G_0 = e_0\left((\mathrm{Aut}\,\mathfrak{g}_0)^0\right)$ and

$$G^* = e^*\left((\mathrm{Aut}\,\mathfrak{g}^*)^0\right)$$

are closed connected semisimple. Moreover σ_0 defines an involution on G_0, σ^* one on G^*; the group $K := G_0 \cap G^*$ is compact and $(G_0^{\sigma_0})^0 \subset K \subset G_0^{\sigma_0}$

$(G^{*\sigma_*})^0 \subset K \subset G^{*\sigma_*}$.

The OSL associated to (G_0, K) is ~~...~~ isomorphic to $(\mathfrak{g}_0, \theta_0)$ and the OSL associated to (G^*, K) is isomorphic to $(\mathfrak{g}^*, \theta^*)$.

Proof: It will turn out to be essential to understand the relation between τ, $\theta^{\mathbb{C}}$, $(\theta^*)^{\mathbb{C}}$ and τ^*, where τ^* denotes the complex conjugation of $\mathfrak{g} = \mathfrak{g}^* + i\mathfrak{g}^*$ wrt $\mathfrak{g}^*$. All these 4 ~~..~~ $\tau, \tau^*, \theta^{\mathbb{C}} = (\theta^*)^{\mathbb{C}}$ are automorphisms of $\mathfrak{g}^{\mathbb{R}}$, that is $\mathfrak{g}$ seen as real Lie algebra.

It will be convenient to present to action of these automorphisms in a fashion:

$\mathfrak{k}$	$i\mathfrak{k}$	$\mathfrak{p}$	$i\mathfrak{p}$	
Id	Id	−Id	−Id	θ^{σ}
Id	Id	−Id	−Id	$\theta^{*\sigma}$
Id	−Id	Id	−Id	τ
Id	−Id	−Id	Id	τ^{*}

~~The all are involutions~~

These automorphisms all have order 2, commute pairwise and:

$$\theta^{\sigma}\tau = \tau^{*} \qquad (\theta^{*})^{\sigma}\tau^{*} = \tau.$$

Thus if $\langle \alpha, \beta \rangle$ denotes the subgroup of $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$ generated by two elements α, β we have:

$$\langle \theta^{\sigma}, \tau \rangle = \langle (\theta^{*})^{\sigma}, \tau^{*} \rangle = \langle \tau, \tau^{*} \rangle.$$

Define then the following automorphisms of the Lie group $Aut(\mathfrak{g}^{\mathbb{R}})$: for $\alpha \in Aut(\mathfrak{g}^{\mathbb{R}})$,

$$t(\alpha) = \tau \alpha \tau^{-1}, \quad t^*(\alpha) = \tau^* \alpha \tau^{*-1}, \quad \Omega(\alpha) = \theta^{\mathbb{C}} \alpha (\theta^{\mathbb{C}})^{-1}$$
$$= (\theta^*)^{\mathbb{C}} \alpha (\theta^{*\mathbb{C}})^{-1}.$$

Now $Aut(\mathfrak{g}) = \{ \alpha \in Aut(\mathfrak{g}^{\mathbb{R}}) :$
$$\alpha(i \cdot Z) = i\alpha(Z)$$
$$\forall Z \in \mathfrak{g} \}$$

is hence a closed subgroup of $Aut(\mathfrak{g}^{\mathbb{R}})$; it is invariant under t : indeed,

$\tau(iZ) = -i\tau(Z) \ \forall Z \in \mathfrak{g}$ and hence if $\alpha \in Aut(\mathfrak{g})$, $\tau \alpha \tau^{-1}(iZ) = \tau\alpha(-i\tau^{-1}(Z))$

$= (\tau(-i)\alpha\tau^{-1}(Z) = i\, \tau\alpha\tau^{-1}(Z).$

Then we claim that the image of

$$e_0 : Aut\, \mathfrak{g}_0 \longrightarrow Aut\, \mathfrak{g}$$
$$\alpha \longmapsto \alpha^{\mathbb{C}}$$

coincides with $(\operatorname{Aut} \mathfrak{g})^{t}$ the t-fixed point subgroup of $\operatorname{Aut} \mathfrak{g}$. We leave this as an easy verification. Thus the image of e_0 is closed and hence

$$e_0 : \operatorname{Aut} \mathfrak{g}_0 \to (\operatorname{Aut} \mathfrak{g})^{t}$$

is a Lie group isomorphism which implies that $G_0 = e((\operatorname{Aut} \mathfrak{g}_0)^{0}) = [(\operatorname{Aut} \mathfrak{g})^{t}]^{0}$ is closed connected semisimple.

The same argument applies to G^{*}.

Now we have the relations in $\operatorname{Aut}(\operatorname{Aut}(\mathfrak{g}^{\mathbb{R}}))$:

t, t^{*}, Ω commute are of order two,

$\Omega t = t^{*}$, $\Omega t^{*} = t$ and:

$\langle \Omega, t \rangle = \langle \Omega, t^{*} \rangle = \langle t, t^{*} \rangle$ are

subgroups of $\operatorname{Aut}(\operatorname{Aut}(\mathfrak{g}^{\mathbb{R}}))$.

Recall that $\sigma_0 = \Omega|_{(\mathrm{Aut}\,g)^t}$

$$\sigma^* = \Omega|_{(\mathrm{Aut}\,g)^{t^*}}.$$

Then:

$$G_0 \cap G^* = [(\mathrm{Aut}\,\underline{g})^t]^0 \cap [(\mathrm{Aut}\,g)^{t^*}]^0$$

is open in

$$(\mathrm{Aut}\,\underline{g})^t \cap (\mathrm{Aut}\,g)^{t^*} = (\mathrm{Aut}\,g)^{\langle t, t^* \rangle}$$

$$= (\mathrm{Aut}\,g)^{\langle t, \Omega \rangle} = (\mathrm{Aut}\,g)^{\langle t^*, \Omega \rangle}.$$

Thus $G_0 \cap G^*$ is open in $[(\mathrm{Aut}\,g)^t]^\Omega$

but it is also contained in $[(\mathrm{Aut}\,g)^t]^0 = G_0$

hence it is open in:

$$[(\mathrm{Aut}\,g)^t]^0 \cap [(\mathrm{Aut}\,g)^t]^\Omega = G_0^\sigma.$$

Thus $(G_0^\sigma)^0 \subset G_0 \cap G^* \subset G_0^\sigma$.

The compactness follows from the compactness of G_0^σ which is a direct consequence of Thm IV.15. $\boxed{\xi}$

<u>Def. IV. 27</u> Let $(\mathfrak{g}_0, \theta_0)$ be a reducible OSL of non-compact type and (G_0, K) (G^*, K) as above. Then we call $M^* = G^*/K$ the compact dual of the symmetric space of non-compact type $M = G_0/K$.

Observe that here K is connected and $G_0 = IS(M)^0$, $G^* = IS(M^*)^0$.

The following remarkable theorem is the beginning of many interesting developments.

<u>Thm IV.28</u>

Let $M = G_0/K$ be a symmetric space of non-compact type and $M^* = G^*/K$ its compact dual. Then there is a canonical isomorphism

$$\Omega^k(M)^{G_0} \cong H^k(M^*, \mathbb{R})$$

where $\Omega^k(M)^{G_0}$ is the space of G_0-invariant smooth differential k-forms on M and $H^k(M^*, \mathbb{R})$ is the singular cohomology of M^* with $\mathbb{R}$-coefficients.

We give a short outline of the proof.

Given a real vectorspace V let $\mathrm{Alt}_k(V)$ be the space of alternating forms

$$V^k \longrightarrow \mathbb{R}$$

in k-variables.

The following is then an easy exercise:

<u>Lemma IV. 29</u> Let M be a symmetric space, $G = \mathrm{Is}(M)^\circ$, $o \in M$, $K = \mathrm{Stab}_G(o)$ and $D_e\pi : \mathfrak{p} \longrightarrow T_oM$ the isomorphism for which $(D_e\pi)\circ \mathrm{Ad}(k) = D_o L_k \circ D_e\pi \quad \forall k \in K$.

Then

$$\Omega^k(M)^G \xrightarrow{\sim} \mathrm{Alt}_k(\mathfrak{p})^{\mathrm{Ad}(K)}$$

The isomorphism is obtained by restricting $\omega \in \Omega^k(M)^G$ to $o \in M$, $\omega_o \in \mathrm{Alt}_k(T_oM)$

and pulling it back $(D_e \pi)^*(\omega_o) \in \mathrm{Alt}_k(\mathfrak{p})$.

The following is an interesting lemma due to E. Cartan:

<u>Lemma IV. 30</u> Let M = symmetric space, $G = \mathrm{Is}(M)^\circ$ and $\omega \in \Omega^k(M)^G$. Then

$$d\omega = 0.$$

<u>Proof</u>: Let $o \in M$, $s_o \in \mathrm{Is}(M)$ the symmetry at o; let $\omega \in \Omega^k(M)^G$. Since $\forall g \in G$, $s_o g s_o \in G$, we have $s_o^* \omega \in \Omega^k(M)^G$. Now it is clear that

$$(s_o^* \omega)_o = (-1)^k \omega_o$$

and since $s_o^* \omega$ and $(-1)^k \omega$ are invariant forms coinciding at o, they coincide everywhere: $s_o^* \omega = (-1)^k \omega$.

Applying d we get:

$$d(s_0^* \omega) = (-1)^k d\omega$$

But $d(s_0^* \omega) = s_0^* d\omega$; since $d\omega$ is an invariant $(k+1)$-form we have $s_0^* d\omega = (-1)^{k+1} d\omega$

which implies $-d\omega = d\omega$ hence $d\omega = 0$.

$\square$

Let now U be a compact connected Lie group and $U \times X \to X$ a smooth action on a smooth manifold X.

<u>Lemma IV.31</u>

The inclusion of complexes

$$\Omega^k(X)^U \longrightarrow \Omega^k(X)$$

induces an isomorphism in cohomology.

Proof: Let $d\mu$ be a Haar measure of total mass 1 on U.

First: let $\alpha \in \Omega^k(X)^U$ and assume that $\alpha = d\beta$ where $\beta \in \Omega^{k-1}(X)$. Since U-action commutes with d we have

$$\alpha = d(u^*\beta) \quad \forall u \in U$$

hence

$$\alpha = \int_U d(u^*\beta)\, d\mu(u) = d\left(\int_U u^*\beta\, d\mu(u)\right)$$

Now observe $\int_U u^*\beta\, d\mu(u) \in \Omega^{k-1}(X)^U$.

This shows injectivity.

Surjectivity: let $\alpha \in \Omega^k(X)$ with $d\alpha = 0$; since U is connected every u is diffeotope to Id and hence α and $u^*\alpha$ represent the same class in $H^k_{dR}(X)$ de Rham cohomology. Thus $\forall\, Z \in H_k(X, \mathbb{R})$

any C^1-cycle Z we have

$$\int_Z \alpha = \int_Z u^*\alpha \qquad \forall u \in \mathcal{V}.$$

By Fubini we conclude

$$\int_Z \alpha = \int_{\mathcal{V}} \Big(\int_Z u^*\alpha\Big)\, d\mu(u)$$

$$= \int_Z \Big(\int_{\mathcal{V}} u^*\alpha \, d\mu(u)\Big)$$

Thus: $\displaystyle\int_Z \Big(\alpha - \int_{\mathcal{V}} u^*\alpha \, d\mu(u)\Big) = 0$

$\forall Z \in H_k(X, \mathbb{R})$ which by de Rham's theorem implies that α and $\displaystyle\int_{\mathcal{V}} u^*\alpha \, d\mu(u)$ represent the same cohomology class.

This shows surjectivity. $\boxtimes$

Proof of Thm IV. 28

We have with $o = eK \in M \cap M^*$:

$$\Omega^k(M)^{G_0} \xrightarrow{\simeq} (\mathrm{Alt}_k(\mathfrak{p}))^{\mathrm{Ad}(K)}$$

Now
$$\mathfrak{p} \longrightarrow i\mathfrak{p}$$
$$X \longmapsto iX$$

is an $\mathrm{Ad}(K)$ - equivariant isomorphism of real vector spaces, hence

$$\mathrm{Alt}_k(\mathfrak{p})^{\mathrm{Ad}(K)} \simeq \mathrm{Alt}_k(i\mathfrak{p})^{\mathrm{Ad}(K)}$$

and the latter is isomorphic to $\Omega^k_{\mathbb{R}}(M^*)^{G^*}$.

Now the inclusion of complexes

since G^* is compact
$$\Omega^k_{\mathbb{R}}(M^*)^{G^*} \longrightarrow \Omega^k(M^*)$$

induces an isom. in cohomology;

but by lemma IV. 30 $(\Omega^k(M^*)^{G^*}, d)$

is equal to its cohomology and hence

$$\Omega^k(M^*)^{G^*} \simeq H^k_{DR}(M^*)$$

the latter being de Rham cohomology which itself is isomorphic to $H^k(M^*; \mathbb{R})$.

$\boxed{\text{Z}}$