Now we show:

<u>Thm V. 3</u>. A symmetric space of non-compact type is CAT(0).

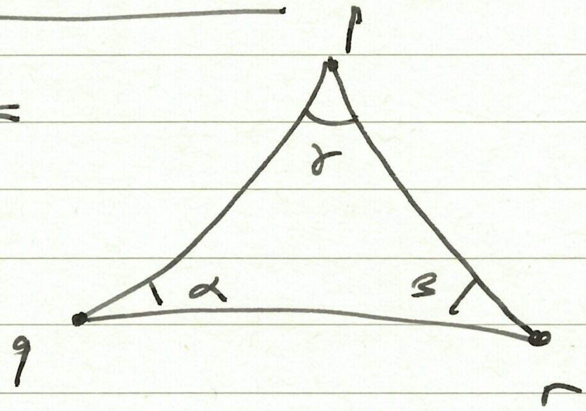
Recall from Remark IV. 14:

<u>Prop. V. 4</u> Let M be SS of non-compact type with associated OIL $(\mathfrak{g}, \theta)$ and assume M is endowed with the $Is(M)^\circ$ - inv. metric coming from restriction to $\mathfrak{p}$ of the Killing form.

Then $\mathrm{Exp}_p : T_p M \longrightarrow M$ is a distance increasing diffeo.

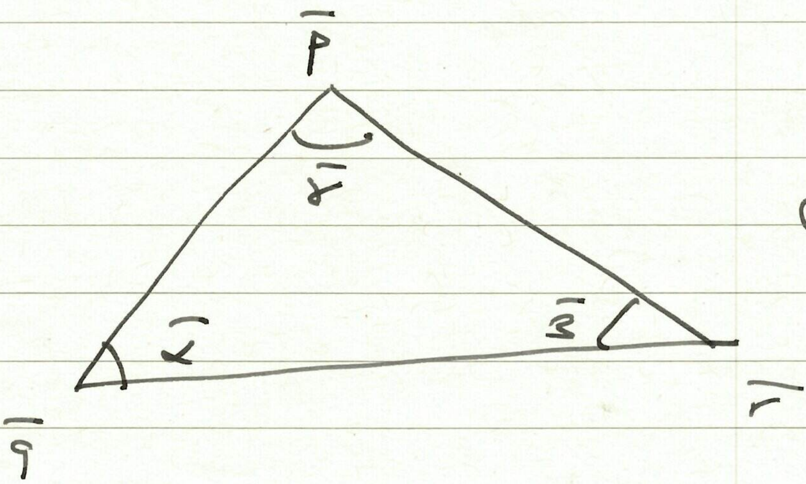
Observe that this implies that M is uniquely geodesic.

Lemma V.5 Let

Δ = 

$\subset M$

and

$\bar{\Delta}$ = 

$\subset \mathbb{E}^2$

comparison triangle. Then:

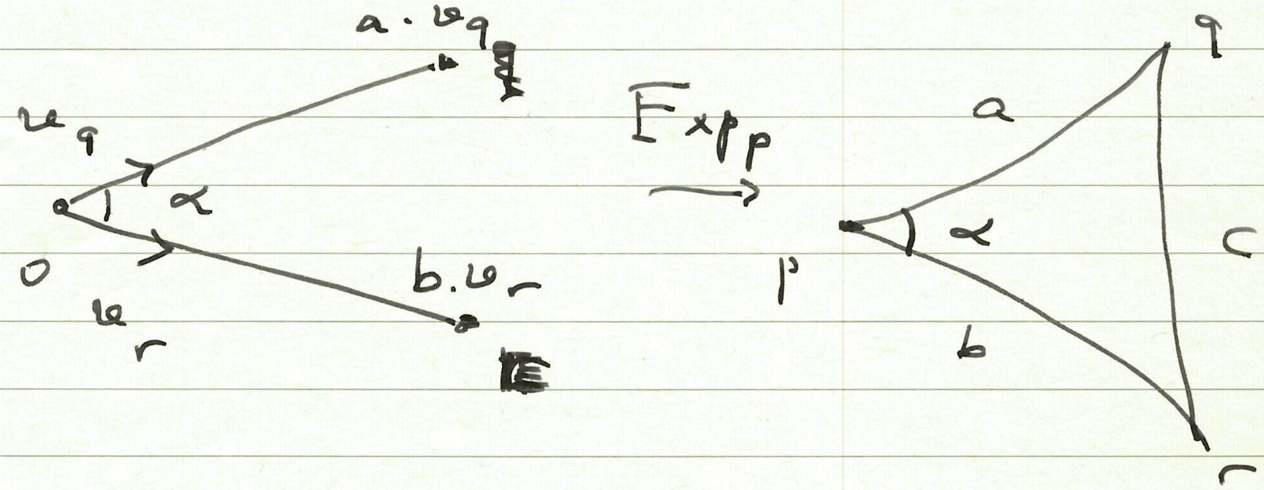
$\alpha \leq \bar{\alpha}$, $\beta \leq \bar{\beta}$, $\gamma \leq \bar{\gamma}$.

Proof: Let thus $p, q, r \in M$ and three geodesic segments $[p,q]$, $[p,r]$, $[r,q]$ in M. Then $[p,q]$ resp.

$[p,r]$ are determined by unit tangent vectors v_q, v_r in T_pM.

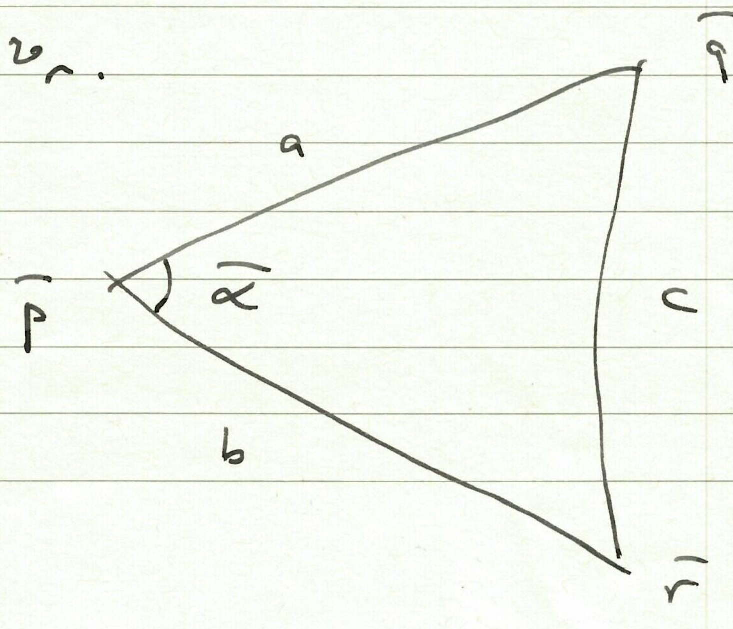
With $a = d(p,q)$, $b = d(p,r)$ will have then:

$$\mathrm{Exp}_p(a v_q) = q, \quad \mathrm{Exp}_p(b v_r) = r:$$



By definition α = angle at o between v_q and v_r.

Let



be the comparison triangle. Then:

$$a^2 + b^2 - 2ab\cos\bar{\alpha} = c^2 = d(q,r)^2$$

$$\geq \|a v_q - b v_r\|^2 = a^2 + b^2 - 2ab\cos\alpha$$

which implies: $\cos\alpha \geq \cos\bar{\alpha}$.

But since $\alpha, \bar{\alpha}$ are in $[0,\pi]$ we get

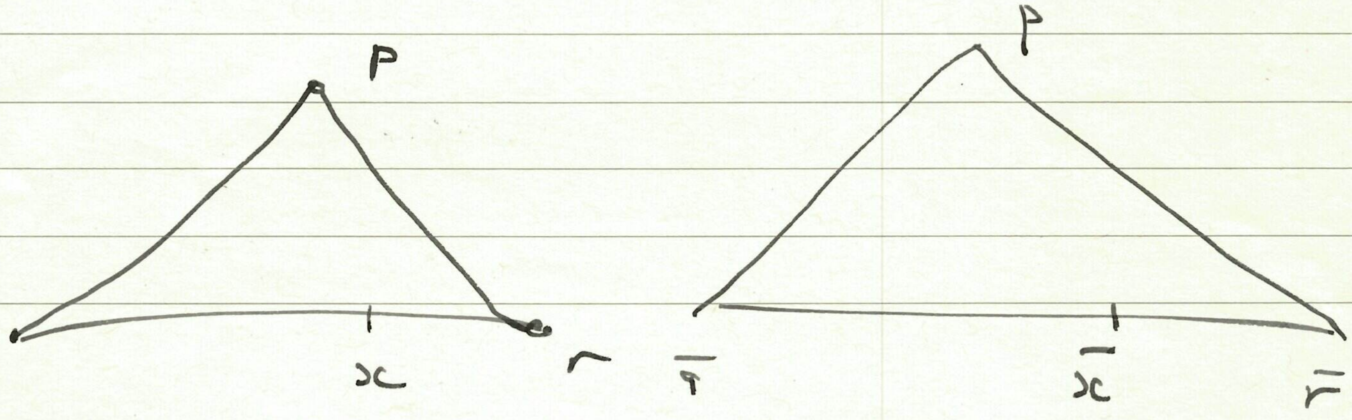
$$\bar{\alpha} \geq \alpha.$$

<u>Proof of Thm V.3.</u>

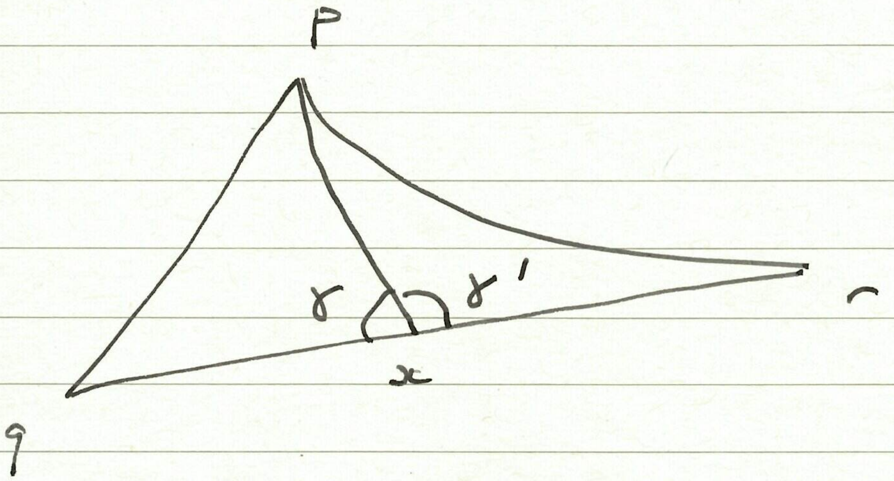
Proceeds in two steps:

(1) Let $\Delta(p,q,r) \subset M$

$\bar{\Delta}(\bar{p},\bar{q},\bar{r}) \subset \mathbb{E}^2$



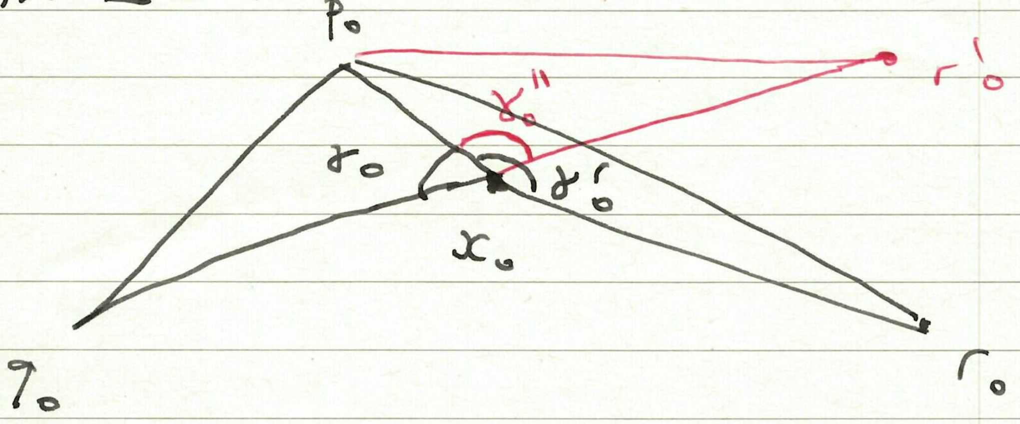
Claim: $d(p,x) \le d(\bar{p},\bar{x})$.



For $\Delta(p,q,x)$ and $\Delta(p,x,r)$

take the corresponding comparison triangles

in $\mathbb{E}^2$:



And observe by lemma V. 5 that:

$$\gamma_0 \ge \gamma, \quad \gamma_0' \ge \gamma' \quad \text{so} \quad \gamma_0 + \gamma_0' \ge \pi.$$

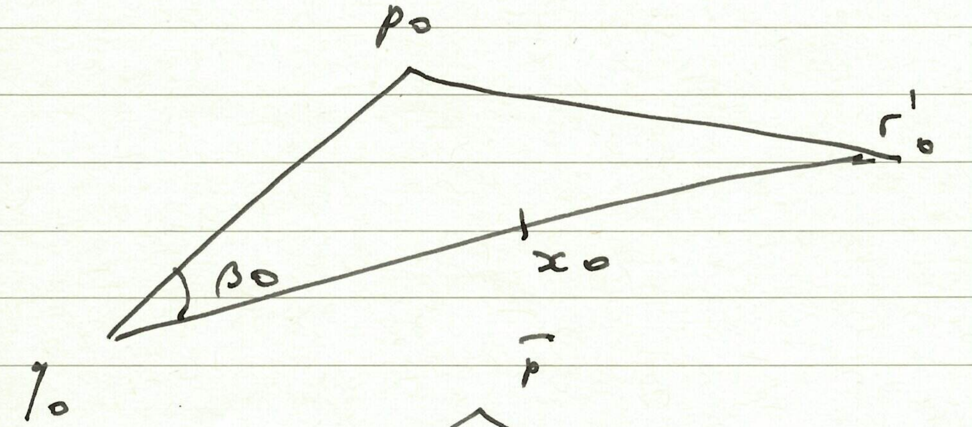
Let q_0, x_0, r_0' be colinear such that

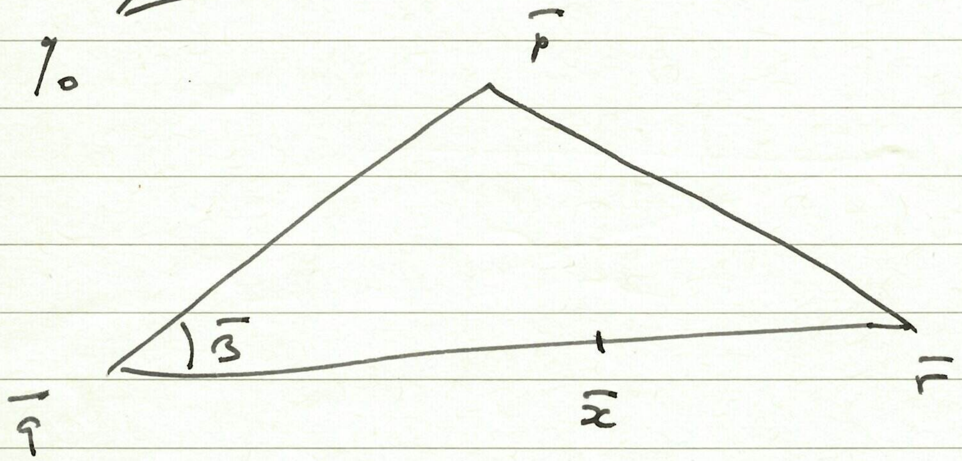
$d(x_0, r_0') = d(x_0, r_0)$ and $\gamma_0'' = \measuredangle(p_0, x_0, r_0')$.

Since $\gamma_0' \geq \gamma_0''$ we conclude from the cosine theorem that $d(p_0, r_0) \geq d(p_0, r_0')$.

This implies:

$$d(\bar{p}, \bar{r}) = d(p_0, r_0) \geq d(p_0, r_0') .$$





Since $d(q_0, p_0) = d(\bar{q}, \bar{p})$, $d(q_0, r_0') = d(\bar{q}, \bar{r})$

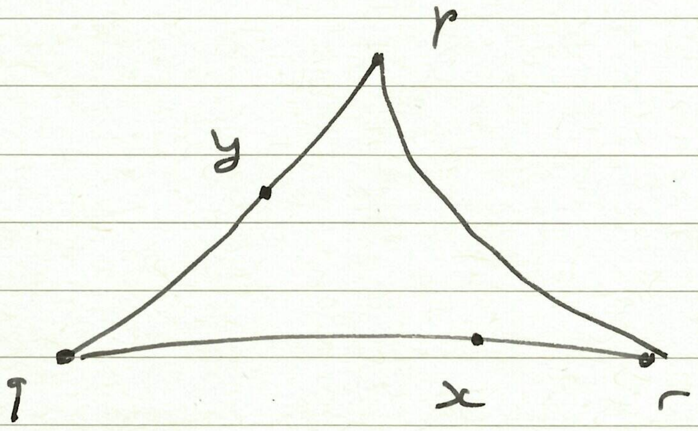
the cosine theorem implies

$$\bar{\beta} \geq \beta_0 .$$

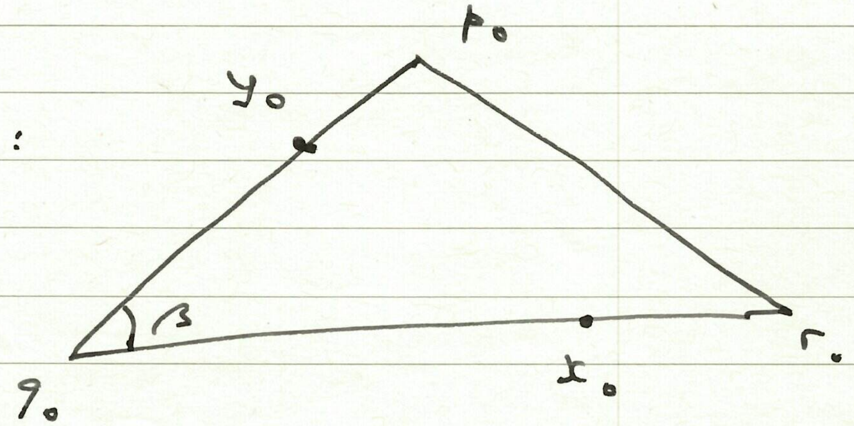
and hence: $d(\bar{p}, \bar{x}) \geq d(p_0, x_0) = d(p, x)$.

(2) Now we complete the proof.

Δ:



Let Δ_0 :



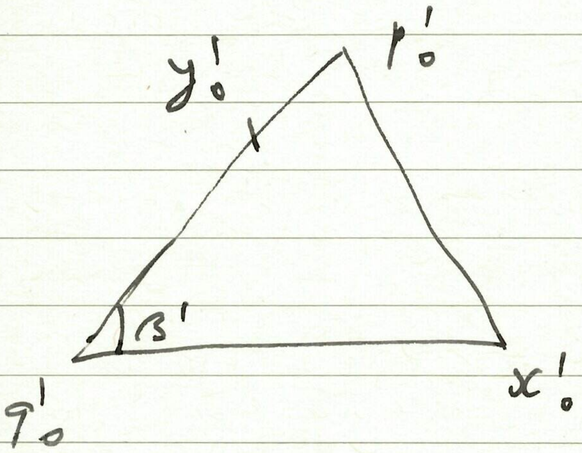
be the comparison triangle of Δ. Then

by (1) : $d(p,x) \leq d(p_0,x_0)$.

Next let

$\Delta_0' = \Delta_0'(p_0', x_0', q_0')$ be the comparison

triangle of $\Delta(p,x,q)$:



Then $d(p_0', x_0') = d(p, x) \leq d(p_0, x_0)$

Since $d(q_0, p_0) = d(q_0', p_0')$ and

$d(q_0, x_0) = d(q_0', x_0')$

the cosine law implies $\beta' \leq \beta$.

Thus:

$$d(y,x) \underset{\substack{\uparrow \\ \text{Claim (1)}}}{\leq} d(y_0', x_0') \leq d(y_0, x_0)$$

$\blacksquare$

Then we obtain:

Thm V. 6.

Let M be a SS of non-compact type and $Is(M)^0 < G < Is(M)$. Let $K = Stab_G(o)$.

(1) K meets every connected component of G. In particular $Is(M)^0$ is of finite index in $Is(M)$.

(2) Any compact subgroup $V \subset G$ has a fixed point in M.

(3) The set $\{ Stab_G(p) : p \in M \}$ coincides with the set of maximal subgroups of G.

Proof:

(1) G/K^0 is a Galois covering of $M = G/K$ with Galois group K/K^0. Since M is diffeomorphic to $T_p M$ in particular simply connected we must have that G/K^0 is diffeomorphic to $M \times K/K^0$ which implies the first assertion.

(2) Let $S = U \cdot o$. Then S is bounded ~~and hence~~ U-invariant and Prop. II.2(i) implies that $c_S \in M$ is U-fixed.

(3) The fact that any compact subgroup of G is contained in some $\mathrm{Stab}_G(p)$ follows from (2). The fact that if $p \neq q$ $\mathrm{Stab}_G(p) \neq \mathrm{Stab}_G(q)$ requires an additional

argument which we sketch here:

If for some $p \neq q$ $\mathrm{Stab}_G(p) = \mathrm{Stab}_G(q)$ then there is $Z \in \mathfrak{p}$, $Z \neq 0$ with $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{z}) Z = 0$. Thus ~~every~~ some irreducible factor of $(\mathfrak{g}, \theta)$ would have the same property, contradicting irreducibility. $\boxed{\Sigma}$