V. 2. Flats and rank.

Let M be a symmetric space.

Def. V.7 A flat subspace in M is a totally geodesic submanifold $F \subset M$ such that $\forall p \in F$ and $\{X, Y\} \subset T_pF$ orthonormal, $\sigma_p(\langle X, Y\rangle) = 0$.

Flat subspaces can be characterised algebraically as follows:

Prop. V.8. Let (G, K) be a RSP either of compact or non-compact type, $M = G/K$, $o \in M$, $(\mathfrak{g}, \theta)$ the associated OSL and $\mathrm{Exp}: \mathfrak{p} \longrightarrow M$, $X \longmapsto \exp X . o$. Then $F \ni o$ is a flat subspace iff

$$F = \mathrm{Exp}\, \mathfrak{a}$$

where $\mathfrak{a} \subset \mathfrak{p}$ is an abelian subspace, that is $[\mathfrak{a}, \mathfrak{a}] = 0$.

If M is of non-compact type and F is flat:
~~In this case~~ $\mathrm{Exp}: \mathfrak{n} \longrightarrow F$ is an isometry.

<u>Proof:</u> Since $F \ni o$ is totally geodesic, $F = \mathrm{Exp}\, \mathfrak{n}$ where $\mathfrak{n} \subset \mathfrak{p}$ is a Lie triple system (Thm II.46); moreover (see Remark V.48) there is a connected subgroup $G' < G$ leaving F invariant and acting transitively. Hence F is flat iff the sectional curvature $\sigma_o \equiv 0$ vanishes at $o \in F$. But if M is of compact type (see Proof of Thm IV.2) we have $\sigma_o(\langle X, Y \rangle) = -B_{\mathfrak{g}}([X,Y],[X,Y])$ $\forall X, Y \in \mathfrak{p}$; thus it vanishes $\forall X, Y \in \mathfrak{n}$ iff $[X,Y] = 0$ $\forall X, Y \in \mathfrak{n}$. If M is of non-compact type the argument is similar.

For the last assertion remember that

$$D_X \mathrm{Exp} = \left(D_0 L_{\exp X}\right)\left(D_e \tilde{\pi}\right)\left(\sum_{n=0}^{\infty} \frac{\left(T_X|_{\mathfrak{p}}\right)^n}{(2n+1)!}\right)$$

where $T_X|_{\mathfrak{p}} = (\mathrm{ad}\, X)^2|_{\mathfrak{p}}$.

Clearly if $X, Y \in \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] = 0$ then

$$D_X \mathrm{Exp}|_{\mathfrak{p}} = \left(D_0 L_{\exp X}\right)\left(D_e \pi\right). \mathrm{Id}$$

and since $D_0 L_{\exp X}$ realises the parallell transport we conclude that $\mathrm{Exp}|_{\mathfrak{p}}$ is ~~an isometry.~~ a Riemannian isometry. Since it is also a distance increasing diffeo from $\mathfrak{p}$ to M we conclude that $\mathrm{Exp}|_{\mathfrak{p}} : \mathfrak{p} \longrightarrow F$ is an isometry. ∎

<u>Def. V.9</u> : Let M be of compact or non-compact type. The rank $rk(M)$ is the maximal dimension of a totally geodesic flat submanifold of M.

Thus the computation of the rank of M reduces to the computation of the dimension of a maximal abelian subspace of $\mathfrak{p}$.
This still can be cumbersome, but we are going to relate these maximal subspaces to so called regular elements. This will be a major tool to understand flat subspaces.

Let $(\mathfrak{g}, \theta)$ be an OSL and $X \in \mathfrak{g}$.
Then $Z_{\mathfrak{g}}(X) := \{ Y \in \mathfrak{g} : [Y, X] \}$ is the centralizer of X in $\mathfrak{g}$, which is a subalgebra of $\mathfrak{g}$. Assume $X \in \mathfrak{p}$;

since θ is an automorphism, $\theta(Z_{\mathfrak{g}}(X)) =$
$= Z_{\mathfrak{g}}(\theta(X)) = Z_{\mathfrak{g}}(-X) = Z_{\mathfrak{g}}(X)$. Hence

$$Z_{\mathfrak{g}}(X) = (Z_{\mathfrak{g}}(X) \cap \mathfrak{k}) + (Z_{\mathfrak{g}}(X) \cap \mathfrak{p}).$$

If now $\mathfrak{a} \subset \mathfrak{p}$ is an abelian subspace with $\mathfrak{a} \ni X$ then $\mathfrak{a} \subset Z_{\mathfrak{g}}(X) \cap \mathfrak{p}$.
Hence $Z_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is maximal abelian so soon as it is abelian.

<u>Def. V.10.</u> $X \in \mathfrak{p}$ is a <u>regular</u> element if $Z_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is abelian.

<u>Thm V. 11.</u> Let M be a SS of compact or non-compact type, $G = I_S(M)^0$, $o \in M$ and $(\mathfrak{g}, \theta)$ the associated OSL.
Let $\mathfrak{a} \subseteq \mathfrak{p}$ be ~~abelian Then~~ maximal abelian. Then $\exists\, X \in \mathfrak{p}$ with

$$\mathfrak{a} = Z_{\mathfrak{g}}(X) \cap \mathfrak{p}.$$

<u>Proof:</u>

(1) Assume M of compact type.
Let $\mathfrak{a} \subset \mathfrak{p}$ be maximal abelian subspace. Then $\mathfrak{a} \subset \mathfrak{g}$ is an abelian subalgebra and hence $\exp(\mathfrak{a}) \subset G$ is a connected abelian subgroup. Let $A := \overline{\exp(\mathfrak{a})}$ be its closure. Then $A \subset G$ is closed and connected; moreover, since $\theta(X) = -X \;\forall X \in \mathfrak{a}$, we have $\sigma(\exp(X)) = (\exp X)^{-1} \;\forall X \in \mathfrak{a}$, hence $\sigma(a) = a^{-1} \;\forall a \in A$. Thus $A = \exp(\mathrm{Lie}\, A)$ with $\mathrm{Lie}\, A \subset \mathfrak{p}$ and abelian; by maximality of $\mathfrak{a}$ this implies $\mathfrak{a} = \mathrm{Lie}\, A$ and hence $A = \exp(\mathfrak{a})$ is closed. Thus A is a Torus and by Kronecker there exists $X \in \mathfrak{a}$

such that $\{\exp tX : t \in \mathbb{R}\}$ is dense in A. If now $Y \in Z_{\mathfrak{g}}(X)$ then $\{\exp sY : s \in \mathbb{R}\}$ commutes with $\{\exp tX : t \in \mathbb{R}\}$ hence with $A = \exp \mathfrak{a}$ and thus $[Y, \mathfrak{a}] = 0$. If moreover $Y \in \mathfrak{p}$ then $\mathfrak{a} + \mathbb{R}Y \subset \mathfrak{p}$ is abelian hence by maximality $Y \in \mathfrak{a}$. This shows $Z_{\mathfrak{g}}(X) \cap \mathfrak{p} = \mathfrak{a}$ and in particular X is regular.

(2) Assume M of non-compact type, $(\mathfrak{g}_0, \theta_0)$ its associated (reduced OSL) and M^* its compact dual (Def IV. 27) with OSL $(\mathfrak{g}^*, \theta^*)$. Then $\mathfrak{a} \subset \mathfrak{p}$ is abelian iff $i\mathfrak{a} \subset i\mathfrak{p}$ is abelian. By the above there exists $iX \in i\mathfrak{p}$ such that

$$Z_{\mathfrak{g}^*}(iX) \cap i\mathfrak{p} = i\mathfrak{a}.$$

But then $Z_{\mathfrak{g}_0}(X) \cap \mathfrak{p} = \mathfrak{a}$. $\boxed{\equiv}$

Let now (G, K) be a symmetric pair of non compact type $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition. Clearly if $\mathfrak{a} \subset \mathfrak{p}$ is maximal abelian so is $Ad_G(k)(\mathfrak{a}) \subset \mathfrak{p}$ $\forall k \in K$. The following is then a central result:

<u>Thm V.12.</u> Let (G, K) be a Riem. symmetric pair of non-compact type and $\mathfrak{a}, \mathfrak{a}'$ in $\mathfrak{p}$ maximal abelian. Then there exists $k \in K$ with

$$\mathfrak{a}' = Ad_G(k)(\mathfrak{a}).$$

<u>Proof:</u> Let H, H' in $\mathfrak{p}$ such that $\mathfrak{a} = Z_{\mathfrak{g}}(H) \cap \mathfrak{p}$ and $\mathfrak{a}' = Z_{\mathfrak{g}}(H') \cap \mathfrak{p}$.

Consider the smooth function

$$K \longrightarrow \mathbb{R}$$
$$k \longmapsto B_{\mathfrak{g}}(Ad(k)H, H').$$

Since K is a compact manifold this function must have a critical point, that is $k_0 \in K$ with:

$$\frac{d}{dt}\Big|_{t=0} B_{\mathfrak{g}}\left(\mathrm{Ad}(k_0 \exp tZ)H, H'\right) = 0$$

$$\forall Z \in \mathfrak{k}.$$

Now $\displaystyle \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(k_0 \exp tZ)H$

$$= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(k_0)\, e^{t\,\mathrm{ad}(Z)} H$$

$$= \mathrm{Ad}(k_0)\,[Z, H]$$

$$= \left[\mathrm{Ad}(k_0)Z, \mathrm{Ad}(k_0)H\right].$$

Thus: $\displaystyle B_{\mathfrak{g}}\left(\left[\mathrm{Ad}(k_0)Z, \mathrm{Ad}(k_0)H\right], H'\right) = 0$

$$= B_{\mathfrak{g}}\left(\mathrm{Ad}(k_0)Z, \left[\mathrm{Ad}(k_0)H, H'\right]\right)$$

But $B_{\mathfrak{g}}\Big|_{\mathfrak{k}\times\mathfrak{k}} \ll 0$ hence

$$[\mathrm{Ad}(k_0) H, H'] = 0.$$

Thus $\mathrm{Ad}(k_0) H \in \mathcal{Z}_{\mathfrak{g}}(H') \cap \mathfrak{p} = \mathfrak{a}'$

Thus $\mathfrak{a}' \subset \underbrace{\mathcal{Z}_{\mathfrak{g}}(\mathrm{Ad}(k_0) H) \cap \mathfrak{p}}$

$$\mathrm{Ad}(k_0)\left[Z_{\mathfrak{g}}(H) \cap \mathfrak{p} \right] = \mathrm{Ad}(k_0)\mathfrak{a}.$$

By maximality $\mathfrak{a}' = \mathrm{Ad}(k_0)\,\mathfrak{a}$. ∎

<u>Corollary V.13</u>. Let $M = G/K$ be a symmetric space of non-compact type, and F, F' maximal flat subspaces. Then $\exists\, g \in G$ with

$$gF = F'.$$

We are now in a position to compute the rank in some examples.

Examples V. 14.

(i) $M = SL(n,\mathbb{R}) / SO(n)$, rank $M = n-1$.

Indeed $\mathfrak{p} = \{ X \in M_n(\mathbb{R}) : \operatorname{tr} X = 0, \; {}^t X = X \}$.

Then: $\mathfrak{a} = \{ X \in \mathfrak{p} : X \text{ diagonal} \}$

is clearly abelian. In addition for

$$H = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}, \quad X \in M_n(\mathbb{R}):$$

$$[H, X]_{ij} = (t_j - t_i) X_{ij}.$$

Which implies that if $t_i - t_j \neq 0 \;\; \forall i \neq j$ then $Z_{\mathfrak{g}}(H) = \mathfrak{a}$. This shows the claim.

(ii) $\operatorname{rk}\left(SO(p,q) / S(O(p) \times O(q)) \right) = \min(p,q)$

(iii) $\operatorname{rk}\left(Sp(2n,\mathbb{R}) / U(n) \right) = n$.

V. 3. Roots and Root Spaces.

Let now $(\mathfrak{g}, \theta)$ be an OSL of non-compact type. Recall that:

$$\langle Z_1, Z_2 \rangle := - B_{\mathfrak{g}}(Z_1, \theta(Z_2))$$

defines a scalar product on $\mathfrak{g}$ such that $\forall X \in \mathfrak{p}$, $\mathrm{ad}(X)$ is symmetric. (lemma IV. 14). If now $\mathfrak{a} \subset \mathfrak{p}$ is an abelian subspace then

$$\{\mathrm{ad}(X) : X \in \mathfrak{a}\}$$

is a family of endomorphisms of $\mathfrak{g}$ that is commuting and symmetric, hence simultaneously diagonalizable. Let $\lambda \in \mathfrak{a}^* = \mathcal{L}(\mathfrak{a}, \mathbb{R})$

and $$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} : \mathrm{ad}(H) X = \lambda(H) X \quad \forall H \in \mathfrak{a}\}.$$

<u>Def V. 15</u> : A root of $\mathfrak{a}$ in $\mathfrak{g}$ is a nonzero linear form $\alpha \in \mathfrak{a}^* \setminus (0)$ such that $\mathfrak{g}_\alpha \neq 0$. Then $\mathfrak{g}_\alpha$ is the associated root space.