V. 3 Roots and root spaces.

Let $(\mathfrak{g}, \theta)$ be an OSL of non-compact type. Recall that

$$\langle Z_1, Z_2 \rangle = - B_{\mathfrak{g}}(Z_1, \theta(Z_2))$$

defines a scalar product on $\mathfrak{g}$ such that (lemma IV. 14) $\forall X \in \mathfrak{p}$, $\mathrm{ad}(X)$ is a symmetric endomorphism w.r.t. $\langle\, ,\, \rangle$.
Thus if $\mathfrak{a} \subset \mathfrak{p}$ is an abelian subspace $\{\mathrm{ad}(X) : X \in \mathfrak{a}\}$ is a commuting family of diagonalizable endomorphisms, hence simultaneously diagonalizable. This motivates the following definition; let $\lambda \in \mathfrak{a}^*$:

$$\mathfrak{g}_\lambda = \{ X \in \mathfrak{g} : \mathrm{ad}(H) X = \lambda(H) X \quad \forall H \in \mathfrak{a} \}.$$

We have the following rather immediate properties:

<u>Lemma V. 15</u>

(1) $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu} \quad \forall \lambda, \mu \in \mathfrak{a}^*$

(2) $\theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda} \quad \forall \lambda \in \mathfrak{a}^*$.

<u>Proof:</u>

(1) follows from the fact that $ad(H)$ is a derivation:

$$ad(H)[X,Y] = [ad(H)X, Y] + [X, ad(H)Y]$$

which by taking $H \in \mathfrak{a}$, $X \in \mathfrak{g}_\lambda$, $Y \in \mathfrak{g}_\mu$ implies the assertion.

(2) follows from $\theta(H) = -H$ for $H \in \mathfrak{a} \subset \mathfrak{p}$ and $\theta\, ad(H)\, \theta^{-1} = ad(\theta(H))$. $\boxed{\Sigma}$

<u>Def. V.16</u> A <u>root</u> of $\mathfrak{a}$ in $\mathfrak{g}$ is a nonzero linear form $\alpha \in \mathfrak{a}^* \setminus (0)$ such that $\mathfrak{g}_\alpha \neq 0$. Then $\mathfrak{g}_\alpha$ is the associated root space.

Let $\Sigma \subset \mathfrak{a}^* \setminus (0)$ be the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$, then it follows from the fact that $\{ ad(H) : H \in \mathfrak{a} \}$ is a commuting family of diagonalizable endomorphisms:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha .$$

Observe that this decomposition is orthogonal wrt the scalar product $\langle \, , \rangle$. Also it follows that Σ is finite. Now let

$$\begin{aligned} \mathfrak{a}^* &\longrightarrow \mathfrak{a} \\ \lambda &\longmapsto H_\lambda \end{aligned}$$

be the isomorphism coming from the

Killing form restricted to $\mathfrak{a}$, namely:

$$\lambda(H) = B(H, H_\lambda) \quad \forall H \in \mathfrak{a}.$$

The central result is then:

Thm V.17 Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and $\Sigma' \subset \mathfrak{a}^* \setminus \{0\}$ the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$. Then Σ' is a root system, that is:

(1) Σ' spans $\mathfrak{a}^*$

(2) $\forall \alpha, \beta \in \Sigma'$:

$$\beta - \frac{2B(H_\alpha, H_\beta)}{B(H_\alpha, H_\alpha)} \alpha \in \Sigma'$$

(3) $$\frac{2B(H_\alpha, H_\beta)}{B(H_\alpha, H_\alpha)} \in \mathbb{Z}.$$

Remark V.18 Here is a geometric interpretation of (2): let us define on $\mathfrak{a}^*$ the scalar product $\langle\ ,\ \rangle$ that corresponds to $B|_{\mathfrak{a}\times\mathfrak{a}}$ via the isomorphism

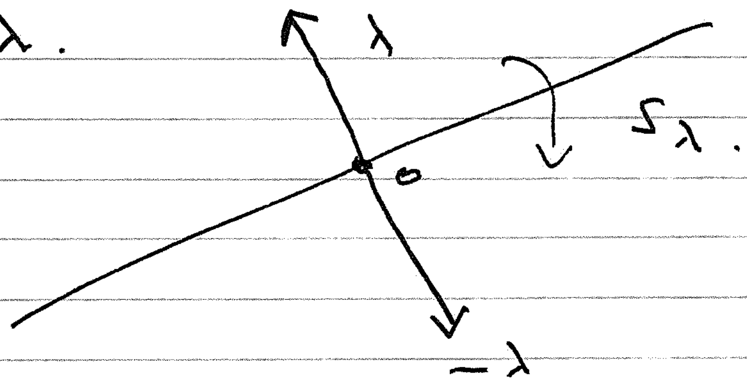
$\mathfrak{a}^* \to \mathfrak{a}$
$\lambda \longmapsto H_\lambda$. Then

$$\frac{2B(H_\alpha, H_\beta)}{B(H_\alpha, H_\alpha)} = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}.$$

Then for every $\lambda \neq 0$ in $\mathfrak{a}^*$,

$$S_\lambda : \mathfrak{a}^* \longrightarrow \mathfrak{a}^*$$
$$\mu \longmapsto \mu - \frac{2\langle\mu,\lambda\rangle}{\langle\lambda,\lambda\rangle}\cdot\lambda$$

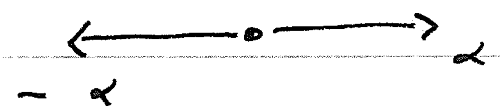
is $\underbrace{\text{the}}_{\text{orthogonal}}$ reflection fixing the hyperplane orthogonal to λ.



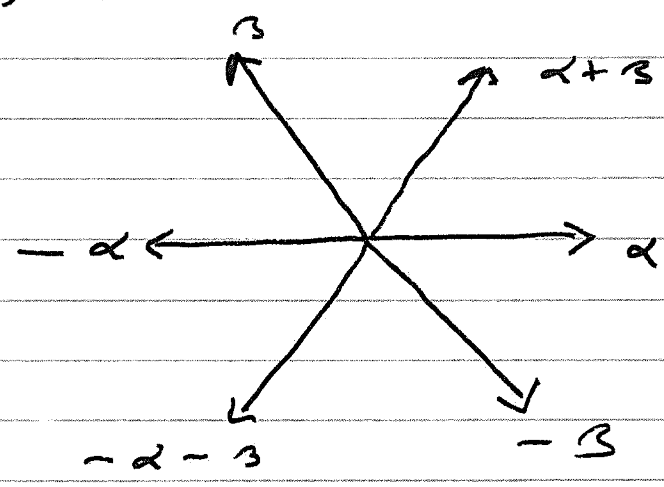
Then (2) reads: $S_\alpha(\Sigma) = \Sigma$ $\forall \alpha \in \Sigma$.

The third property is more mysterious and will be a consequence of the classification of finite dimensional representations of $sl(2,\mathbb{R})$.

<u>Exercise V.19</u> Work out that the root system for $sl(2,\mathbb{R})$ is:



and $sl(3,\mathbb{R})$:



From now on we assume that $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$. Then we observe

Lemma V. 20 $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$.

Proof: $\mathfrak{g}_0 \cap \mathfrak{p} = \{ X \in \mathfrak{p} : [H, X] = 0 \quad \forall H \in \mathfrak{a} \}$.

Thus if $X \in \mathfrak{g}_0 \cap \mathfrak{p}$, $\mathfrak{a} + \mathbb{R} X$ is an abelian subspace which by maximality of $\mathfrak{a}$ implies $X \in \mathfrak{a}$. $\blacksquare$

The central computation is then:

Lemma V. 21. Let $\alpha \in \Sigma'$. Then

$$\forall X \in \mathfrak{g}_\alpha, \quad [X, \theta(X)] = B(X, \theta(X)) H_\alpha$$

$$= - \langle X, X \rangle H_\alpha$$

Proof: The second equality is just the definition of the scalar product on $\mathfrak{g}$.

Let $X \in \mathfrak{g}_\alpha$: applying lemma V. 15 (2) and (1) we get $\theta(X) \in \mathfrak{g}_{-\alpha}$ hence

$$[X, \theta(X)] \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{g}_0 .$$

Next we observe:

$$\theta([X, \theta(X)]) = [\theta(X), X] = -[X, \theta(X)]$$

which implies $[X, \theta(X)] \in \mathfrak{p}$ and hence $[X, \theta(X)] \in \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$.

Now we compute $\forall Y \in \mathfrak{a}$:

$$\langle [X, \theta(X)], Y \rangle = -B_{\mathfrak{g}}([X, \theta(X)], \theta(Y))$$

$$= B_{\mathfrak{g}}([X, \theta(X)], Y) = -B_{\mathfrak{g}}(\theta(X), [X, Y])$$

$$= B_{\mathfrak{g}}(\theta(X), X)\, \alpha(Y) \quad , \text{since } [Y, X] = \alpha(Y) X.$$

$$= B_{\mathfrak{g}}(\theta(X), X)\, B_{\mathfrak{g}}(H_\alpha, Y)$$

$$= B_{\mathfrak{g}}(\theta(X), X) \langle H_\alpha, Y \rangle .$$

That is: $[X, \theta(X)] - B_{\mathfrak{g}}(\theta(X), X)\, H_\alpha$ is in $\mathfrak{a}$ and orthogonal to all of $\mathfrak{a}$, hence zero. $\square$

Define $h_\alpha := \dfrac{2 H_\alpha}{B(H_\alpha, H_\alpha)}$

We deduce:

<u>Lemma V.22.</u> Let $X \in \mathfrak{g}_\alpha$, $X \neq 0$. Let $x_\alpha \in \mathbb{R}_{>0} X$ be the unique vector with $\langle x_\alpha, x_\alpha \rangle = \dfrac{2}{B(H_\alpha, H_\alpha)}$

and $y_\alpha := -\theta(x_\alpha)$. Then:

$$[h_\alpha, x_\alpha] = 2 x_\alpha$$

$$[h_\alpha, y_\alpha] = -2 y_\alpha$$

$$[x_\alpha, y_\alpha] = h_\alpha.$$

<u>Proof</u>: These are really straightforward verifications:

$[h_\alpha, x_\alpha] = \alpha(h_\alpha) x_\alpha$ and by definition

$$\alpha(h_\alpha) = \frac{2\alpha(H_\alpha)}{B(H_\alpha, H_\alpha)} = 2.$$

Similarly: $[h_\alpha, y_\alpha] = -2y_\alpha$.

The last one uses lemma V. 21:

$$[x_\alpha, y_\alpha] = -[x_\alpha, \theta(x_\alpha)]$$

$$= \langle x_\alpha, x_\alpha \rangle H_\alpha = \frac{2H_\alpha}{B(H_\alpha, H_\alpha)} = h_\alpha.$$

$\boxed{\Sigma}$

For $(\mathfrak{sl}(2,\mathbb{R}), \theta)$ with $\theta(x) = -{}^t x$

we choose $\mathfrak{a} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}$

as maximal abelian subspace of $\mathfrak{p}$.

Since $\left[\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix}, \begin{pmatrix}0 & \mu\\ 0 & 0\end{pmatrix}\right] = 2\lambda \begin{pmatrix}0 & \mu\\ 0 & 0\end{pmatrix}$

$$\left[\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix}, \begin{pmatrix}0 & 0\\ \mu & 0\end{pmatrix}\right] = -2\lambda \begin{pmatrix}0 & 0\\ \mu & 0\end{pmatrix}$$

defining $\alpha \in \mathfrak{a}^*$ by $\alpha\left(\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix}\right) = 2\lambda$

we get

$$\mathfrak{g}_\alpha = \left\{ \begin{pmatrix}0 & \mu\\ 0 & 0\end{pmatrix} : \mu \in \mathbb{R} \right\}$$

$$\mathfrak{g}_{-\alpha} = \left\{ \begin{pmatrix}0 & 0\\ \mu & 0\end{pmatrix} : \mu \in \mathbb{R} \right\}$$

$$\mathfrak{g}_0 = \mathfrak{a}.$$

Moreover $B_{\mathfrak{g}}\left(\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix}, \begin{pmatrix}\mu & 0\\ 0 & -\mu\end{pmatrix}\right) = 8\lambda\cdot\mu$

Hence $H_\alpha = \begin{pmatrix}1/4 & 0\\ 0 & -1/4\end{pmatrix}$, $B(H_\alpha, H_\alpha) = \frac{1}{2}$

and $h_\alpha = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

Now a computation gives:

$$< \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} > = 4 = \frac{2}{\frac{1}{2}} = \frac{2}{B(H_\alpha, H_\alpha)}.$$

Thus $x_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

In fact the standard notation for $sl(2, \mathbb{R})$

is $e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

<u>Corollary V. 23</u> Given $X \in \mathfrak{g}_\alpha$, $X \neq 0$, $h_\alpha, x_\alpha, y_\alpha$ as in lemma V. 22 the linear map $sl(2, \mathbb{R}) \to \mathfrak{g}$ defined on a basis by:

$$e_+ \to x_\alpha$$
$$h \to h_\alpha$$
$$e_- \to y_\alpha$$

is an injective Lie algebra homomorphism with image $\underbrace{\mathbb{R} x_\alpha + \mathbb{R} h_\alpha + \mathbb{R} y_\alpha}_{:= sl(2,\mathbb{R})_X} \subset \mathfrak{g}$.

<u>Proof:</u> The linear map $\rho : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$ defined by $\rho(e_+) = x_\alpha$, $\rho(h) = h_\alpha$, $\rho(e_-) = y_\alpha$ sends a basis of $\mathfrak{sl}(2,\mathbb{R})$ to a basis of $\mathfrak{sl}(2,\mathbb{R})_X$, hence is a vector space isomorphism. Next we have by construction:

$$\rho([h, e_+]) = \rho(2e_+) = 2\rho(e_+) = 2x_\alpha$$

$$= [h_\alpha, x_\alpha] = [\rho(h), \rho(e_+)].$$

In the same way one verifies:

$\rho([h, e_-]) = [\rho(h), \rho(e_-)]$ and $\rho([e_+, e_-]) =$

$= [\rho(e_+), \rho(e_-)]$.

By bilinearity of the bracket this implies

$$\rho([x, y]) = [\rho(x), \rho(y)] \quad \forall x, y \in \mathfrak{sl}(2,\mathbb{R}).$$

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