4.3 Roots and root spaces.

Let \((g, \Theta)\) be an OSL of non-compact type. Recall that

\[
\langle z_1, z_2 \rangle = -\beta_g (z_1, \Theta(z_2))
\]

defines a scalar product on \(g\) such that (Lemma IV.14) \(\forall x \in p, \text{ad}(x)\)
is a symmetric endomorphism w.r.t. \(\langle, \rangle\).

Thus if \(p \in g\) is an abelian subspace

\[
\text{ad}(x): x \in p \rightarrow \text{ is a commuting family of diagonalizable endomorphisms, hence simultaneously diagonalizable. }
\]

This motivates the following definition; let

\[
\lambda \in \mathfrak{a}^*:
\]

\[
\mathfrak{a}_{\lambda} = \{ x \in \mathfrak{g} : \text{ad}(\lambda)(x) = \lambda(\lambda) x \ \forall \lambda \in \mathfrak{a}^* \}.
\]
We have the following rather immediate properties:

**Lemma I. 15**

1. \([\mathcal{L}_\lambda, \mathcal{G}_\mu] = \Theta_{\lambda+\mu} \quad \forall \lambda, \mu \in \mathfrak{a}^*\)

2. \(\Theta(\mathcal{G}_\lambda) = \mathcal{G}_{-\lambda} \quad \forall \lambda \in \mathfrak{a}^*\).

**Proof:**

(1) follows from the fact that \(\text{ad}(\mathcal{H})\) is a derivation:

\[
\text{ad}(\mathcal{H}) [X, Y] = [\text{ad}(\mathcal{H}) X, Y] + [X, \text{ad}(\mathcal{H}) Y]
\]

which by taking \(\mathcal{H} \in \mathfrak{a}, \quad X \in \mathcal{G}_\lambda, \quad Y \in \mathcal{G}_\mu\) implies the assertion.

(2) follows from \(\Theta(\mathcal{H}) = -\mathcal{H}\) for \(\mathcal{H} \in \mathfrak{a}\) and \(\Theta \circ \text{ad}(\mathcal{H}) = -\Theta(\mathcal{H})\). \(\Box\)
Def. V.16 A root of \( \alpha \) in \( \mathfrak{g} \) is a nonzero linear form \( \gamma \in \mathfrak{a}^* \setminus \{0\} \) such that \( \gamma(\alpha) = 0 \). Then \( \gamma_\alpha \) is the associated root space.

Let \( \Sigma \subset \mathfrak{a}^* \setminus \{0\} \) be the set of roots of \( \alpha \) in \( \mathfrak{g} \), then it follows from the fact that \( \mathfrak{g} = \mathfrak{ad}(H) \): Here \( \mathfrak{g} \) is a commuting family of diagonalizable endomorphisms:

\[
\mathfrak{g} = \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha.
\]

Observe that this decomposition is orthogonal with the scalar product \( \langle \cdot, \cdot \rangle \). Also it follows that \( \mathfrak{g}' \) is finite. Now let \( \phi^* : \mathfrak{a}^* \rightarrow \mathfrak{a} \)

\[
\phi^* \quad \text{and} \\
2 \quad \mapsto \\
1 \quad \mapsto \text{H}_\alpha
\]

be the isomorphism coming from the
The Killing form restricted to \( \mathfrak{a} \), namely:

\[ \lambda (\mathfrak{a}) = \mathbf{B} (H, H) \quad \forall \mathbf{H} \in \mathfrak{a}. \]

The central result is then:

**Theorem 1.** Let \( \mathfrak{a} \subset \mathfrak{m} \) be a maximal abelian subspace and \( \Sigma = \alpha^* \setminus \{0\} \) the set of roots of \( \mathfrak{a} \) in \( g \). Then \( \Sigma \) is a root system, that is:

1. \( \Sigma' \) spans \( \mathfrak{a}^* \)

2. \( \forall \alpha, \beta \in \Sigma' \):

\[ \beta - \frac{2 \mathbf{B} (\alpha, H_\beta)}{\beta (H_\alpha, H_\alpha)} \alpha \in \Sigma' \]

3. \( \frac{2 \mathbf{B} (\alpha, H_\beta)}{\beta (H_\alpha, H_\alpha)} \in \mathbb{Z} \)

**Remark.** Here is a geometric interpretation of (2): let us define
on $\alpha^*$ the scalar product $\langle \cdot, \cdot \rangle$ that corresponds to $B_1 \mid \alpha \times \alpha$ via the isomorphism $\alpha^* \rightarrow \alpha$. Then

$$\lambda \rightarrow H_2 \quad \text{and} \quad \frac{B(H_2, H_3)}{B(H_2, H_2)} = \frac{2 \langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}.$$

Then for every $x \neq 0$ in $\alpha^*$,

$$I : \alpha^* \rightarrow \alpha^*$$

$$\lambda \rightarrow \lambda - \frac{2 \langle \mu, \lambda \rangle}{\langle \lambda, \lambda \rangle} \cdot \lambda$$

is the reflection fixing the hyperplane orthogonal to $\alpha$.

Then (2) reads: $S_{\alpha}(\Sigma) = \Sigma^{-1}$ for all $\alpha \in \Sigma$.

The third property is more mysterious and
Theorem 38

will be a consequence of the classification of finite dimensional representations of \( sl(2, \mathbb{R}) \).

**Exercise:** Work out that the root system for \( sl(2, \mathbb{R}) \) is:

\[ \begin{align*}
-\alpha & \quad \rightarrow \quad \alpha \\
-\alpha & \quad \rightarrow \quad \alpha \\
-\alpha & \quad \rightarrow \quad \alpha \\
-\alpha & \quad \rightarrow \quad \alpha 
\end{align*} \]

and \( sl(3, \mathbb{R}) \):

\[ \begin{align*}
-\alpha & \quad \rightarrow \quad \alpha \\
-\alpha & \quad \rightarrow \quad \alpha \\
-\alpha & \quad \rightarrow \quad \alpha \\
-\alpha & \quad \rightarrow \quad \alpha 
\end{align*} \]
From now on we assume that $\mathcal{R}$ is maximal abelian in $\mathcal{P}$. Then we observe

\textit{Lemma V.20} \hspace{1cm} \mathcal{G}_0 \cap \mathcal{P} = \mathcal{R}.

\underline{Proof:} \hspace{1cm} \mathcal{G}_0 \cap \mathcal{P} = \left\{ x \in \mathcal{P} : [H, x] = 0 \quad \forall H \in \mathcal{R} \right\}.

Thus if $x \in \mathcal{G}_0 \cap \mathcal{P}$, $a + R \cdot x$ is an abelian subspace which by maximality of \mathcal{R} implies $x \in \mathcal{R}$. \hfill \Box

The central computation is then:

\textit{Lemma V.21.} Let $x \in \Sigma$. Then

\[ \forall x \in \mathcal{G}_x, \quad [x, \Theta(x)] = R \cdot (x, \Theta(x)) H \cdot \frac{1}{2} = - \langle x, x \rangle H \cdot \frac{1}{2} \]
Proof: The second equality is just the definition of the scalar product on \( g \).

Let \( x \in g_x \); applying lemma \( \text{V. 15 (2)} \) and (11) we get \( \Theta (x) \in g_{-x} \), hence

\[
[x, \Theta(x)] \in [g_x, g_{-x}] \subseteq g_0.
\]

Next we observe:

\[
\Theta ([x, \Theta(x)]) = [\Theta(x_1), x] = -[x, \Theta(x)]
\]

which implies

\[
[x, \Theta(x)] \in \mathbb{P}
\]

and hence

\[
[x, \Theta(x)] \in g_0 \cap \mathbb{P} = 02.
\]

Now we compute \( \forall \gamma \in \mathfrak{g} : \)

\[
< [x, \Theta(x)], \gamma > = -\beta_\gamma ([x, \Theta(x)], \Theta(\gamma))
\]

\[
= \beta_\gamma ([x, \Theta(x)], \gamma) = -\beta_\gamma (\Theta(x), [x, \gamma])
\]

\[
= \beta_\gamma (\Theta(x), x) \alpha (\gamma), \text{ since } [\gamma, x] = \alpha (\gamma) x,
\]

\[
= \beta_\gamma (\Theta(x), x) \beta_\gamma (H_x, \gamma)
\]

\[
= \beta_\gamma (\Theta(x), x) < H_x, \gamma >.
\]
That is: $[x, \theta(x)] - \nabla \theta(x_1, x) h x$

is in $\mathcal{A}$ and orthogonal to all of $\mathcal{A}$, hence zero.

Define $h x : = \frac{2 H x}{\beta (H x, H x)}$

We deduce:

Lemma I. II. Let $x \in \mathcal{A}$, $x \neq 0$. Let $x_\alpha \in \mathbb{R}^n, \alpha$ be the unique vector with $\langle x_\alpha, x \rangle = \frac{2}{\beta (H x, H x)}$

and $y_\alpha := - \theta (x_\alpha)$. Then:

$[h x, x_\alpha] = 2 x_\alpha$

$[h x, y_\alpha] = - 2 y_\alpha$

$[x_\alpha, y_\alpha] = h x$.
Proof: These are really straightforward verifications:
\[
[h_2, x_2] = \alpha(h_2)x_2 \quad \text{and by definition} \quad \alpha(h_2) = \frac{2 \mathbf{X}(H_2)}{B(H_2, H_2)} = 2.
\]
Similarly:
\[
[h_2, y_2] = -2y_2.
\]
The last one uses Lemma V. 21:
\[
[x_2, y_2] = -[x_2, \Theta(x_2)] = \frac{2Hx}{B(H_2, H_2)} = h_2.
\]

For \((0, e, \theta, 0)\) with \(\theta(x_1) = -tx_1\)
we choose \(\alpha = \{ (\lambda, 0) : \lambda \in \mathbb{R} \} \) as maximal abelian subspace of \(H\).
\[-5-43-\]

Since \[
\begin{pmatrix}
\lambda & 0 \\
0 & \mu
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
\mu & \lambda
\end{pmatrix} = 2 \lambda \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \mu
\end{pmatrix}
\begin{pmatrix}
\mu & 0 \\
0 & \lambda
\end{pmatrix} = -2 \lambda \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

defining \( \alpha \in \mathfrak{g}^* \) by \( \alpha \left( \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \right) = 2 \lambda \)
we get

\[
\mathfrak{g}_\alpha = \left\{ \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix} : \mu \in \mathbb{R} \right\}
\]

\[
\mathfrak{g}_\lambda = \left\{ \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix} : \mu \in \mathbb{R} \right\}
\]

\[
\mathfrak{g}_0 = \mathfrak{g}
\]

Moreover \( B_\gamma \left( \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} \right) = 8 \lambda \mu \)

Hence \( H_\alpha = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \), \( B(H_\alpha, H_\alpha) = \frac{1}{2} \)

and \( h_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

Now a computation gives:
\[ \langle (0, 1), (0, 1) \rangle = 4 = \frac{2}{1} = \beta(12, 12) \]

Thus \( x_2 = (0, 1), \ y_2 = (1, 0) \).

In fact the standard notation for \( \mathfrak{sl}(2, \mathbb{R}) \)

\[ e_+ = (0, 1), \ e_- = (1, 0), \]

\[ h = (1, 0) \].

**Corollary IV.23** Given \( X \in \mathfrak{g} \), \( X \not= 0 \),

\( h, x, y \) as in lemma IV.22 the

linear map \( \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{g} \) defined

on a basis by:

\[ e_+ \rightarrow x_2 \]

\[ h \rightarrow h_x \]

\[ e_- \rightarrow y_2 \]

is an \( \mathfrak{lie} \) algebra homomorphism

with image \( \mathbb{R} x_2 + \mathbb{R} h_x + \mathbb{R} y_2 \subset \mathfrak{g} \).

\[ : = \mathfrak{sl}(2, \mathbb{R}) \times \]
Proof: The linear map \( f : \text{sl}(2, \mathbb{C}) \to g \)
defined by \( f(e_+) = x_2, f(h) = h_1, f(e_-) = y \) sends a basis of \( \text{sl}(2, \mathbb{C}) \) to a basis of \( \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R}) \), hence is a vector space isomorphism. Next we have by construction:
\[
f([h, e_+]) = f(2e_+) = 2f(e_+) = 2x_2
\]
\[
= [h_2, x_2] = [f(h_1), f(e_+)].
\]
In the same way one verifies:
\[
f([h, e_-]) = [f(h_1), f(e_-)] \quad \text{and} \quad f([e_+, e_-]) =
\]
\[
= [f(e_+), f(e_-)].
\]
By bilinearity of the bracket this implies
\[
f([x, y]) = [f(x_1), f(y_1)] \forall x, y \in \text{sl}(2, \mathbb{C}).
\]