Thus by Corollary V.23 we obtain $\forall X \in \mathfrak{g} \setminus 0$, a representation of $sl(2,\mathbb{K})$ ~~on~~ on $\mathfrak{g}$ via:

$$sl(2,\mathbb{K}) \longrightarrow sl(2,\mathbb{K})_X \xrightarrow{ad} gl(\mathfrak{g}).$$

It is thus essential to understand the representation theory of the Lie algebra $sl(2,\mathbb{K})$. We summarize the relevant information in the following

Thm V.24.

(1) Every finite dimensional representation of $sl(2,\mathbb{K})$ is a direct sum of irreducible ones.

(2) Up to isomorphism every irreducible finite dimensional representation is classified by its dimension. If $\rho : sl(2,\mathbb{K}) \to$
$\to gl(V)$ is an irreducible representation

$\rho(h)$ is diagonalizable with simple eigenvalues

$$1-n,\ 3-n,\ \ldots\ldots,\ n-3,\ n-1$$

where $n = \dim V$.

Examples V. 25

(1) The trivial representation:

$$\rho : \mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{gl}(\mathbb{R})$$
$$X \longmapsto 0$$

$$\rho(h) = 0 = 1-1.$$

(2) The standard representation

$$\rho : \mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{gl}(\mathbb{R}^2)$$
$$X \longmapsto X$$

~~(3) The adjoint~~

The eigenvalues of $\rho(h)$ are $-1, 1$ with eigenspaces $\mathbb{R}e_1$, $\mathbb{R}e_2$.

(3) The adjoint representation:

$$\mathrm{ad} : \mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{gl}(\mathfrak{sl}(2,\mathbb{R}))$$
$$X \longmapsto \mathrm{ad}(X)$$

The eigenvalues of $\rho(h)$ are

$$-2, 0, 2$$

with corr. eigenspaces

$$\mathbb{R}e_-, \mathbb{R}h, \mathbb{R}e_+.$$

(4) The general irreducible representation of dimension $n+1$ of $\mathfrak{sl}(2,\mathbb{R})$ can be described as follows.

Let V_n = space of homogeneous polynomials in (X,Y) of degree n

$$= \left\{ \sum_{k=0}^{n} a_k X^k Y^{n-k} : a_k \in \mathbb{R} \right\}$$

Then the Lie group $SL(2,\mathbb{R})$ acts in V_n by linear substitution:

$$(\rho(g)P)(X,Y) = P((X,Y)g).$$

The representation of $sl(2,\mathbb{R})$ is then its derivative:

$$(D_e\rho)(A)P(X,Y) = (D_{(X,Y)}P)((X,Y)\cdot A).$$

Explicitely: $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$.

$$(D_e\rho)(A)P(X,Y) = \frac{\partial P}{\partial X}(X,Y)(aX+cY) + $$
$$+ \frac{\partial P}{\partial Y}(X,Y)(bX-aY).$$

In particular:

$$(D_e\rho)(h)X^kY^{n-k} = (2k-n)X^kY^{n-k}$$

Thus the eigenvalues are

$$-n, -n+2, \dots, n$$

with corresponding eigenspaces

$$\mathbb{R}Y^n, \mathbb{R}XY^{n-1}, \dots, \mathbb{R}X^{n-1}Y, \mathbb{R}X^n.$$

Observe $\dim V_n = n+1$.

Now let $\alpha, \beta \in \Sigma$ be roots and define

$$W_{\beta,\alpha} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha} \subset \mathfrak{g}.$$

Pick $x \in \mathfrak{g}_\alpha \setminus (0)$ and consider $\mathrm{ad}_\mathfrak{g}$ restricted to $\mathfrak{sl}(2, \mathbb{R})_x = \mathbb{R}x_\alpha + \mathbb{R}h_\alpha + \mathbb{R}y_\alpha$.

Since $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$ we have that $W_{\beta,\alpha}$ is invariant under $\mathrm{ad}_\mathfrak{g}(\mathfrak{sl}(2,\mathbb{R})_x)$.

The eigenvalues of $\mathrm{ad}_\mathfrak{g}(h_\alpha)$ in $W_{\beta,\alpha}$ are precisely:

$$\{\beta(h_\alpha) + 2 \cdot k \;:\; \forall k \in \mathbb{Z} \text{ s.t. } \mathfrak{g}_{\beta + k\alpha} \neq 0\}$$

$$= \{\beta(h_\alpha) + 2k \;:\; \forall k \in \mathbb{Z} \text{ s.t. } \beta + k\alpha \in \Sigma \cup \{0\}\}.$$

Thus let

$$s = \max\{k : \beta + k\alpha \in \Sigma \cup \{0\}\}$$

$$r = \min\{k : \beta + k\alpha \in \Sigma \cup \{0\}\}$$

and let n_X be the maximal dimension of an irreducible $sl(2,\mathbb{R})_X$ submodule in $W_{\beta,\alpha}$ then Thm V.24(2) implies

$$n_X - 1 = \beta(h_\alpha) + 2 \cdot s$$

$$1 - n_X = \beta(h_\alpha) + 2 \cdot r.$$

Which implies that

$$-\beta(h_\alpha) = s + r$$

Which shows assertion (3) in Thm V.17.

Next, since $\alpha(h_\alpha) = 2$, Thm V.24(2) implies that for all $r \leq k \leq s$:

$$\mathfrak{g}_{\beta + k\alpha} \neq 0.$$

As a result since $r \leq 0 \leq s$

we have

$$r \leq -\beta(h_\alpha) \leq s$$

hence $$\mathfrak{g}_{\beta - \beta(h_\alpha)\alpha} \neq 0 .$$

Then either $\beta - \beta(h_\alpha)\alpha \neq 0$ hence

$$\beta - \beta(h_\alpha)\alpha \in \Sigma'$$

or : $\beta = \beta(h_\alpha)\alpha$ which evaluating on

h_α gives $\beta(h_\alpha) = 2\beta(h_\alpha)$ hence $\beta(h_\alpha) = 0$

and $\beta = \beta - \beta(h_\alpha)\alpha \in \Sigma'$.

This shows assertion (2) in Thm V.17.

In fact in the course of the proof we

have shown the following fact of independent

interest :

<u>Lemma V.26</u>. Let $\alpha, \beta \in \Sigma'$ and assume $\beta \notin \mathbb{Z}\cdot\alpha$. Let

$$s = \max\{k \in \mathbb{Z} : \beta + k\alpha \in \Sigma\}$$

$$r = \min\{k \in \mathbb{Z} : \beta + k\alpha \in \Sigma\}$$

then: $\quad \beta + k\alpha \in \Sigma' \quad \forall k \in [r, s] \cap \mathbb{Z}$.

V. 4. Abstract root systems.

Thm V.17 puts in the forefront a structure called root system that leads to a finite reflection group, its Weyl group; the latter is an example of a much wider class of groups called Coxeter groups that acquired prominent status in the theory of buildings and in geometric group theory.

In this section we want to establish certain fundamental properties of root systems and their Weyl group.

Let then $\mathbb{E}$ be an euclidean space with scalar product $\langle\,,\,\rangle$; recall that the reflexion determined by $\gamma \in \mathbb{E}\setminus(0)$ is

$$\sigma_\gamma(\alpha) := \alpha - \frac{2\langle\alpha,\gamma\rangle}{\langle\gamma,\gamma\rangle}\cdot\gamma$$

Def. V.27. A root system of rank $l = \dim E$ is a subset $\Sigma \subset E \setminus \{0\}$ s.t.

(R1) Σ spans E

(R2) $\sigma_\alpha(\Sigma) = \Sigma$ $\forall \alpha \in \Sigma$

(R3) $2\dfrac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ $\forall \alpha, \beta \in \Sigma$.

The system is reduced if in addition

(R4) $\forall \alpha \in \Sigma$, $\mathbb{R}\cdot\alpha \cap \Sigma = \{\alpha, -\alpha\}$.

From R3 we deduce that if $\beta = \lambda \alpha$ $\lambda \neq 0$ then

$$2\lambda = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

and $$\frac{2}{\lambda} = \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$$

which implies $\lambda \in \{\pm\frac{1}{2}, \pm 2, \pm 1\}$.

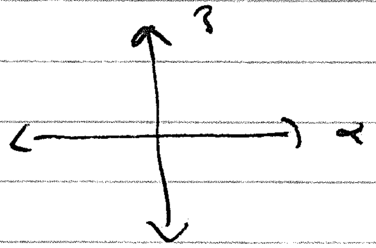
Given a general root system Σ_1 one obtains a reduced one by setting:

$$\Sigma_1' = \{\alpha \in \Sigma_1 : \tfrac{\alpha}{2} \notin \Sigma_1\}.$$

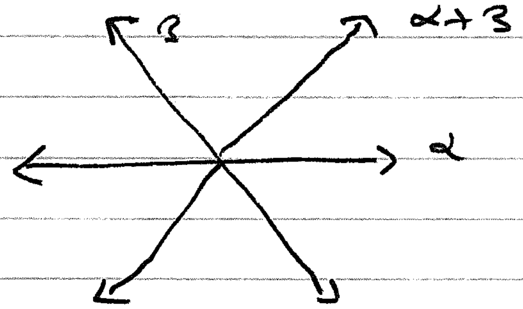
Since our focus will be on the group generated by the σ_α, $\alpha \in \Sigma_1$, we may reduce ourselves to Σ_1' since the reflections in elements of Σ_1' generate the same group.

<u>Examples in rank 2</u>:

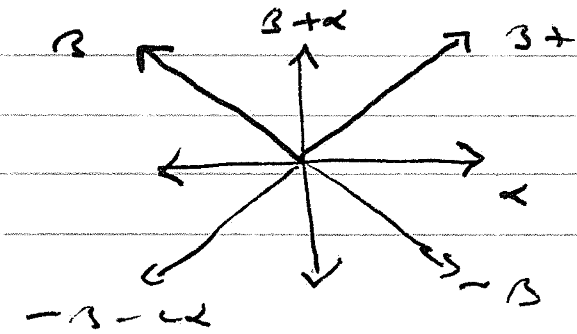
$A_1 \times A_1$



A_2



B_2



The condition R3 constrains the possible angles between two roots. Indeed if $\theta = \measuredangle(\alpha,\beta)$

$$(\beta,\alpha) := \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = \frac{2\|\beta\|}{\|\alpha\|}\cos\theta \in \mathbb{Z}$$

$$(\alpha,\beta) := \frac{2\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle} = \frac{2\|\alpha\|}{\|\beta\|}\cos\theta \in \mathbb{Z}$$

Hence: $\quad 4\cos^2\theta \in \mathbb{Z}$.

and (α,β) and (β,α) have the same sign.

Assuming $\beta \neq \pm\alpha$ the possible values for (α,β), (β,α), θ are assuming $\|\beta\| \geq \|\alpha\|$:

(α,β)	(β,α)	θ	$\frac{\|\beta\|^2}{\|\alpha\|^2} = \frac{(\beta,\alpha)}{(\alpha,\beta)}$
0	0	$\pi/2$	undef.
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Out of this table we get

Lemma V.28 Assume α, β are not proportional roots and $\langle \alpha, \beta \rangle > 0$, then $\beta - \alpha \in \Sigma'$. If $\langle \alpha, \beta \rangle < 0$ then $\beta + \alpha \in \Sigma'$.

Next we turn to the study of basis of a root system.

Def. V.29 A basis of Σ is a subset $\Delta \subset \Sigma$ such that

(1) Δ is a basis of E

(2) ~~to~~ for every $\beta \in \Sigma'$ all coordinates of β in the basis Δ are integers and all of the same sign.

The elements of Δ are called simple roots; the roots ~~in Σ~~ with positive coordinates are called positive, and the others negative. Let Σ^+ be the set of positive roots, then

$$\Sigma = \Sigma^+ \cup (-\Sigma^+).$$

notation: $\Sigma^- = -\Sigma^+$.

<u>Lemma V.30</u> For every $\alpha \neq \beta$ in Δ:

$$\langle \alpha, \beta \rangle \leq 0.$$

<u>Proof:</u> otherwise $\langle \alpha, \beta \rangle > 0$ and by V.28:

$$\gamma = \alpha - \beta \in \Sigma$$

which contradicts property (2) of a basis. $\boxed{\xi}$

Let for every $\gamma \in E \setminus (0)$,

$$P_\gamma = \{ \alpha \in E : \langle \alpha, \gamma \rangle = 0 \}.$$

Then P_γ is a hyperplane and disconnects E into 2 components.

<u>Def V.31</u> (1) A Weyl chamber is a connected component of $E \setminus \left(\bigcup_{\alpha \in \Sigma} P_\alpha \right)$.

(2) $\gamma \in E$ is regular if $\langle \alpha, \gamma \rangle \neq 0$ $\forall \alpha \in \Sigma$; that is γ belongs to some Weyl chamber.

Given $\gamma \in E$ regular let

$$\Sigma_1^+(\gamma) := \{\alpha \in \Sigma_1 : \langle \alpha, \gamma \rangle > 0\}.$$

Clearly $\Sigma_1 = \Sigma_1^+(\gamma) \sqcup (-\Sigma_1^+(\gamma))$.

Call $\alpha \in \Sigma_1^+(\gamma)$ indecomposable if it is not the <u>sum</u> of two elements of $\Sigma_1^+(\gamma)$. Let

$$\Delta(\gamma) = \{\alpha \in \Sigma_1^+(\gamma) : \alpha \text{ is indecomposable}\}.$$

<u>Thm V.32</u> $\forall \gamma \in E$ regular, $\Delta(\gamma)$ is a basis of Σ_1 and any basis is of this form.

<u>Proof:</u>

(1) Every element in $\Sigma^+(\gamma)$ is a $\mathbb{Z}_{\geq 0}$-linear combination of elements in $\Delta(\gamma)$.

By contradiction: take α a counterexample with $\langle \alpha, \gamma \rangle$ minimal.

Then α must be decomposable, that is

$$\alpha = \beta_1 + \beta_2 \quad \text{with } \beta_i \in \Sigma^+(\gamma).$$

But then $\langle \gamma, \alpha \rangle = \langle \gamma, \beta_1 \rangle + \langle \gamma, \beta_2 \rangle$

and since $0 < \langle \gamma, \beta_i \rangle \implies \langle \gamma, \beta_j \rangle < \langle \gamma, \alpha \rangle$.

Hence both β_1, β_2 are a $\mathbb{Z}_{\geq 0}$ – linear combination of elements in $\Delta(\gamma)$ and so is α.

(2) If $\alpha \neq \beta$ in $\Delta(\gamma)$ then $\langle \alpha, \beta \rangle \leq 0$.

Otherwise $\alpha - \beta$ and $\beta - \alpha$ are roots hence WLOG $\alpha - \beta \in \Sigma^+(\gamma)$; but then

$$\alpha = (\alpha - \beta) + \beta$$

contradicting the hypothesis that α is indecomposable.

(3) $\Delta(\gamma)$ is linearly independent.

~~Otherwise $\Delta(\gamma) = A \sqcup B$~~

let $\sum_{\alpha \in \Delta(\gamma)} \lambda_\alpha \alpha = 0$ be a linear relation. ~~Partition $\Delta(\gamma) = A$~~

Let $A = \{\alpha \in \Delta(\gamma) : \lambda_\alpha > 0\}$

$B = \{\beta \in \Delta(\gamma) : \lambda_\beta < 0\}$.

Then $$\sum_{\alpha \in A} \lambda_\alpha \alpha = \sum_{\beta \in B} (-\lambda_\beta) \beta .$$

Thus $$\Big\| \sum_{\alpha \in A} \lambda_\alpha \alpha \Big\|^2 = \sum_{\substack{\alpha \in A \\ \beta \in B}} \lambda_\alpha (-\lambda_\beta) \langle \alpha, \beta \rangle$$

$$\leq 0$$

which implies

$$\sum_{\alpha \in A} \lambda_\alpha \alpha = 0 = \sum_{\beta \in B} (-\lambda_\beta) \beta .$$

But then $$\sum_{\alpha \in A} \lambda_\alpha \underbrace{\langle \alpha, \gamma \rangle}_{> 0} = 0$$

which implies $A = \phi$ and $B = \phi$.

(4) It follows that $\Delta(\gamma)$ is a basis of Σ_1.

(5) Let now $\Delta \subset \Sigma_1$ be any basis.
Since Δ is a basis of E there exists $\gamma \in E$ with $\langle \alpha, \gamma \rangle > 0$. $\forall \alpha \in \Delta$. (exercise).
Then clearly γ is regular and:

$$\Sigma_1^+ \subset \Sigma^+(\gamma) \quad \text{and} \quad \Sigma_1^- \subset \Sigma_1^-(\gamma)$$

Thus we have equality in both cases.
Thus $\Sigma_1^+ = \Sigma_1^+(\gamma)$ and $\Delta^{(\gamma)}$ consists of indecomposable elements $\Rightarrow$ $\Delta(\gamma) \subset \Delta$
Hence $\Delta(\gamma) = \Delta$.

$\boxed{\xi}$

Let now $W :=$ the subgroup of $O(E)$ the orthogonal group of E generated by the set $\{\sigma_\alpha : \alpha \in \Sigma_1\}$ of all reflections.

Then $w(\Sigma') = \Sigma' \quad \forall w \in W$ and

$$w\sigma_\alpha w^{-1} = \sigma_{w(\alpha)} \quad \forall w \in W \quad \forall \alpha \in \Sigma_1'.$$

As a result W permutes the set of Weyl chambers; it also permutes the set of basis of Σ'.

Now if C is a Weyl chamber and $\gamma, \gamma' \in C$ then $\Sigma_1^+(\gamma) = \Sigma_1^+(\gamma')$ hence $\Delta(\gamma) = \Delta(\gamma')$.

This gives a W-equivariant bijection between the set of Weyl chambers and the set of basis of Σ_1'.

<u>Thm V.33</u> Let Δ be a basis of Σ'.

(1) W is generated by $\{\sigma_\alpha : \alpha \in \Delta\}$.

(2) W acts simply transit. on the set of bases.

(3) W acts simply transit. on the set of Weyl chambers.

Lemma V.34 $\forall \alpha \in \Delta$: σ_α permutes $\Sigma_1^+ \setminus \{\alpha\}$

Proof: Let $\beta \in \Sigma_1^+ \setminus \{\alpha\}$. Then there is $\gamma \in \Delta$ such that the γ-coordinate in β is > 0; now $\sigma_\alpha(\beta) = \beta - (\beta, \alpha)\alpha$ and the coord. of $\sigma_\alpha(\beta)$ wrt γ is unchanged. Hence $\sigma_\alpha(\beta) \in \Sigma_1^+$. $\boxed{\xi}$

The following is then immediate:

Lemma V.35 Let $\delta = \sum_{\beta \in \Sigma_1^+} \beta$. Then

$$\sigma_\alpha(\delta) = \delta - 2\alpha \quad \forall \alpha \in \Delta.$$

Lemma V.36. Let $\alpha_1, \dots, \alpha_n \in \Delta$, and write $\sigma_i = \sigma_{\alpha_i}$. If $\sigma_1 \cdots \sigma_{n-1}(\alpha_n) \in \Sigma_1^+$ then there is $1 \leq i < n$ with

$$\sigma_1 \cdots \sigma_n = \sigma_1 \cdots \hat{\sigma}_i \cdots \sigma_{n-1}.$$

<u>Proof:</u> We write the hypothesis as

$$\sigma_1 \cdots \sigma_{n-1}(\alpha_n) \in \Sigma_1^-$$

since $\sigma_n(\alpha_n) = -\alpha_n$.

Now consider the sequence

$$\sigma_{n-1}(\alpha_n)$$

$$\sigma_{n-2}\sigma_{n-1}(\alpha_n)$$

$$\vdots$$

$$\sigma_1 \cdots \sigma_{n-1}(\alpha_n) \in \Sigma_1^-.$$

If $\sigma_{n-1}(\alpha_n) \in \Sigma_1^-$ then we must have

$\alpha_{n-1} = \alpha_n$ and hence $\sigma_1 \cdots \sigma_n = \sigma_1 \cdots \sigma_{n-2}$.

Otherwise $\sigma_{n-1}(\alpha_n) \in \Sigma_1^+$; consider

then $1 \leq i \leq n-2$ minimal such that

$$\sigma_{i+1} \cdots \sigma_{n-1}(\alpha_n) \in \Sigma_1^+$$

$$\sigma_i \sigma_{i+1} \cdots \sigma_{n-1}(\alpha_n) \in \Sigma_1^-.$$

Then we must have

$$\underbrace{\sigma_{i+1} \cdots \sigma_{n-1}}_{w}(\alpha_n) = \alpha_i$$

Hence $\sigma_i = \sigma_{\alpha_i} = \sigma_{w(\alpha_n)} = w\sigma_{\alpha_n}w^{-1} = w\sigma_n w^{-1}$

that is:

$$w\sigma_n = \sigma_i w$$

or

$$\sigma_{i+1} \cdots \sigma_{n-1}\sigma_n = \sigma_i\sigma_{i+1} \cdots \sigma_{n-1}.$$

Multiplying on both sides by $\sigma_1 \cdots \sigma_i$ we get

$$\sigma_1 \cdots \sigma_n = \sigma_1 \cdots \sigma_{i-1}\sigma_{i+1} \cdots \sigma_{n-1}.$$

∎

We get immediately:

<u>Lemma V.37.</u> Let $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_n}$ where n is the minimal ~~such that σ is~~ number of factors in a product decomp. of σ as simple reflections. Then:

$$\sigma(\alpha_n) \in \Sigma^-.$$

<u>Proof of Thm V.33</u>.

Let W' be the subgroup of W generated by $\{\sigma_\alpha : \alpha \in \Delta\}$.

(1) Let $\gamma \in E$ be a regular element. We show that $\exists\, \sigma \in W'$ such that

$$\langle \sigma(\gamma), \alpha \rangle > 0 \quad \forall \alpha \in \Delta .$$

This will imply that if C is the Weyl chamber containing γ, $\sigma(C)$ is the Weyl chamber associated to the basis Δ.

To do this and let $\delta := \sum_{\alpha \in \Sigma^+} \alpha$ and choose $\sigma^* \in W'$ with $\langle \sigma(\gamma), \delta \rangle$ is maximal. Then $\forall \alpha \in \Delta$:

$$\langle \sigma(\gamma), \delta \rangle \geq \langle \sigma_\alpha \sigma(\gamma), \delta \rangle = \langle \sigma(\gamma), \sigma_\alpha \delta \rangle$$

$$= \langle \sigma(\gamma), \delta - 2\alpha \rangle$$

which implies $\langle \sigma(\gamma), \alpha \rangle \geq 0 \quad \forall \alpha \in \Delta$

and hence $\langle \sigma(\gamma), \alpha \rangle > 0 \quad \forall \alpha \in \Delta$ since

$\sigma(\gamma)$ is regular as well.

This shows that W' acts transitively

on Weyl chambers hence on basis.

(2) Given $\alpha \in \Sigma_1$ $\exists\, \sigma \in W'$ with $\sigma(\alpha) \in \Delta$.

Since W' acts transitively on the set of

basis this amounts to show that any $\alpha \in \Sigma_1$

is element of some basis. But let

now $\gamma \in P_\alpha \setminus \left(\bigcup_{\beta \in \Sigma} P_\beta \right)$ and pick

γ' close to γ with $\langle \gamma', \alpha \rangle = \varepsilon > 0$ and

$|\langle \gamma', \beta \rangle| > \varepsilon. \ \forall \beta \in \Sigma^!$
~ finite

Then $\alpha \in \Sigma^+(\gamma')$ and α is indecomposable.

(3) $W = W'$.

By (2) $\exists\, \sigma \in W'$ with $\sigma(\alpha) \in \Delta$; that

is $\sigma_\alpha = \sigma\, \sigma_{\sigma(\alpha)}\, \sigma^{-1} \in W'$.

(4) W acts freely on the set of basis.

Let $\sigma(\Delta) = \Delta$. Let $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_n}$

minimal product of simple reflections.

Then $\sigma(\alpha_n) \in \Delta \subset \Sigma^+$ a contradiction.

∎