Thus by Corollary I.23 we obtain \( V \times \mathfrak{g}_{\psi} \),
a representation of \( \mathfrak{sl}(2,\mathbb{C}) \) on \( \mathfrak{g} \) via:

\[
\text{ad}
\mathfrak{sl}(2,\mathbb{C}) \rightarrow \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{gl}(\mathbb{C}).
\]

It is thus essential to understand the representation theory of the Lie algebra
\( \mathfrak{sl}(2,\mathbb{C}) \). We summarize the relevant information in the following

Thm I.24.

(1) Every finite dimensional representation
of \( \mathfrak{sl}(2,\mathbb{C}) \) is a direct sum of irreducible ones.

(2) Up to isomorphism every irreducible
finite dimensional representation is
classified by its dimension. If \( \phi : \mathfrak{sl}(2,\mathbb{C}) \rightarrow \mathfrak{gl}(V) \) is an irreducible representation
$f(h)$ is diagonalizable with simple eigenvalues

$$1 - n, 3 - n, \ldots, n - 3, n - 1$$

where $n = \dim V$.

**Examples II.25**

1. The trivial representation:

$$f : SL(2, \mathbb{R}) \to gl(1 \mathbb{R})$$

$$x \mapsto 0$$

$$f(h) = 0 = 1 - 1.$$  

2. The standard representation

$$f : SL(2, \mathbb{R}) \to gl(2 \mathbb{R})$$

$$x \mapsto x$$

3. The antipodal

The eigenvalues of $f(h)$ are $-1, 1$ with eigenspace $\mathbb{R}e_1, \mathbb{R}e_2$.  

(3) The adjoint representation:

\[ \text{ad} : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(\mathfrak{sl}(2,\mathbb{R})) \]

\[ x \mapsto \text{ad}(x) \]

The eigenvalues of \( \varphi(h) \) are:

\[ -2, 0, 2 \]

with corresponding eigenspaces:

\[ \mathbb{R} e_{-}, \mathbb{R} h, \mathbb{R} e_{+} \]

(4) The general irreducible representation of dimension \( n+1 \) of \( \mathfrak{sl}(2,\mathbb{R}) \) can be described as follows.

Let \( V_{n} = \text{space of homogeneous polynomials in } (x, y) \) of degree \( n \)

\[ V_{n} = \left\{ \sum_{n=0}^{n} \alpha_{k} x^{k} y^{n-k} : \alpha_{k} \in \mathbb{R} \right\} \]

Then the Lie group \( SL(2,\mathbb{R}) \) acts in \( V_{n} \) by linear substitutions.
The representation of \( \sigma \) is given by its derivate:

\[
\left( D_{\infty} \right) \sigma (x,y) = \left( D_{x} \right) \sigma (x,y).
\]

Explicitly:

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

\[
\left( D_{\infty} \right) \sigma (x,y) = \frac{2p}{2x} (x,y) (ax+cy) + \frac{2p}{2y} (x,y) (by-ax).
\]

In particular:

\[
\left( D_{\infty} \right) (h) X^r Y^s = (2n-r) X^r Y^s.
\]

Thus the eigenvalues are

\[-n, -n+2, \ldots, n\]

with corresponding eigenspaces

\[RY^r, RXY^{r-1}, \ldots, RX^{n-1}Y, RX^n.\]

Observe \( \dim V_n = n+1 \).
Now let \( \alpha, \beta \in \Sigma \) be roots and define
\[
\mathcal{N} = \bigoplus_{\beta, \alpha \in \Sigma} \mathfrak{g} \beta + \mathfrak{g} \alpha.
\]

Pick \( x \in \mathfrak{g}_2 \setminus \{0\} \) and consider \( \text{ad}_g \) restricted to \( \mathfrak{sl}(2, \mathbb{R}) \) as \( \mathfrak{sl}_2 + \mathfrak{h}_2 + \mathfrak{r}_2 \).

Since \( x_2 \in \mathfrak{g}_2 \), \( y_2 \in \mathfrak{g}_2 \) we have that \( \mathcal{N}_2 \) is invariant under \( \text{ad}_g \) \( \mathfrak{sl}(2, \mathbb{R}) \).

The eigenvalues of \( \text{ad}_g \) \( (h_2) \) in \( \mathcal{N}_2 \) are precisely:
\[
\left\{ \beta(h_2) + 2 \cdot k : 4k \in \mathbb{Z} \text{ and } \beta \neq 0 \right\}
\]
\[
= \left\{ \beta(h_2) + 2 \cdot k : k \in \mathbb{Z} \text{ and } \beta + k \cdot 2 \in \Sigma \right\}
\]
Thus let
\[
\beta = \max \{ k : \beta + k \in \Sigma \cup \{ 0 \} \}
\]
\[
\gamma = \min \{ k : \beta + k \in \Sigma \cup \{ 0 \} \}
\]
and let \( n \) be the maximal dimension of an irreducible \( \text{SL}(2, \mathbb{K}) \times \text{sl}(n, \mathbb{K}) \) sub-module in \( W \), then Thm IV.24(2) implies
\[
\begin{align*}
\kappa & = \beta(h_x) + 2 \cdot \gamma \\
1 - \kappa & = \beta(h_x) + 2 \cdot \gamma.
\end{align*}
\]
Which implies that
\[
-\beta(h_x) = \Delta + \gamma
\]
which shows condition (3) in Thm IV.17.

Next, since \( \kappa(h_x) = 2 \), Thm IV.24(2) implies that for all \( \gamma \leq h_x \leq 2 \cdot \gamma \):
\[
\beta \neq 0.
\]
As a result since \( r \leq 0 \leq s \)
we have

\[
-\Delta \leq -\beta(h_\omega) \leq \Delta
\]

hence

\[
\gamma \to 0.
\]

Then either \( \beta - \beta(h_\omega) \to \) hence

\[
\beta - \beta(h_\omega) \in \Sigma',
\]

or

\[
\beta = \beta(h_\omega) \to \text{ which evaluating on}
\]

\( h_2 \)

\[
\beta(h_2) = 2 \beta(h_\omega) \to \text{hence } \beta(h_\omega) = 0
\]

and

\[
\beta = \beta - \beta(h_\omega) \to \in \Sigma'.
\]

This shows assertion (2) in Thm IV.(f).

In fact in the course of the proof we have shown the following fact of independent interest:
Lemma 5.26. Let \( \alpha, \beta \in \Sigma \) and assume \( \beta \notin \Sigma^* \). Let

\[
\beta = \max \left\{ k \in \mathbb{Z} : \beta + k \alpha \in \Sigma \right\}
\]

or

\[
\beta = \min \left\{ k \in \mathbb{Z} : \beta + k \alpha \in \Sigma \right\}
\]

Then: \( \beta + k \alpha \in \Sigma \) \( \forall k \in [0, \Delta] \cup \mathbb{Z} \).
II, 4. Abstract root systems.

Thm II. 17 puts in the forefront a structure called root system that leads to a finite reflection group, i.e., Weyl group; the latter is an example of a much wider class of groups called Coxeter groups that acquired prominent status in the theory of buildings and in geometric group theory.

In this section we want to establish certain fundamental properties of root systems and their Weyl group.

Let then $E$ be an euclidean space with scalar product $\langle \cdot , \cdot \rangle$, recall that the reflection determined by $x \in E \setminus \{0\}$ is

$$
\sigma_x (\xi) := \xi - \frac{2 \langle x, \xi \rangle}{\langle x, x \rangle} \cdot x
$$
Def. II. 27. A root system of rank $l = \dim \mathfrak{E}$ is a subset $\Sigma \subseteq \mathfrak{E} \setminus \{0\}$ s.t.

(R1) $\Sigma$ spans $\mathfrak{E}$

(R2) $a_\alpha(E) = E \quad \forall \alpha \in \Sigma$

(R3) $2 \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \quad \forall \beta, \alpha \in \Sigma$.

The system is reduced if in addition

(R4) $\forall \alpha \in \Sigma$, $R \cdot \alpha \cap \Sigma = \{\alpha, -\alpha\}$.

From R3 we deduce that if $\beta = \lambda \alpha$

$\lambda \neq 0$ then

$2 \lambda = \frac{2 \langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$

and

$\frac{2}{\lambda} = \frac{2 \langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$

which implies

$\lambda \in \left\{ \pm \frac{1}{2}, \pm 2, \pm 1 \right\}$. 
Given a general root system \( \Sigma \) one obtains a reduced one by setting:
\[
\Sigma' = \{ \alpha \in \Sigma : \frac{\alpha}{\| \alpha \|} \in \Sigma' \}
\]
Since our focus will be on the group generated by the \( \Sigma' \), \( \alpha \in \Sigma' \), we may reduce ourselves to \( \Sigma' \) since the reflections in elements of \( \Sigma' \) generate the same group.

Examples in rank 2:

\( A_1 \times A_1 \)

\( A_2 \)

\( B_2 \)
The condition $k^2$ constrains the possible angles between two roots. Indeed, if $\alpha = x(\xi, \eta)$

\[
\frac{2 \langle \xi, \xi \rangle}{\langle \xi, x \rangle} = \frac{2|x|^2}{1 \times 1} \cos \theta \in \mathbb{Z}
\]

\[
\frac{2 \langle \eta, \eta \rangle}{\langle \eta, x \rangle} = \frac{2|\eta|^2}{1 \times 1} \cos \theta \in \mathbb{Z}
\]

Hence: $4 \cos \theta \in \mathbb{Z}$.

and $(\xi, \xi)$ and $(\eta, \eta)$ have the same sign.

Assuming $\beta \neq \xi$ the possible value for $(\xi, \xi)$, $(\eta, \eta)$, $\theta$ are assuming $|\lambda_0| > 1|x|y|z|:

<table>
<thead>
<tr>
<th>$(\xi, \xi)$</th>
<th>$(\eta, \eta)$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$\pi/2$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$\pi/3$</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>$2\pi/3$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$\pi/4$</td>
</tr>
<tr>
<td>-1</td>
<td>-2</td>
<td>$3\pi/4$</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>$\pi/6$</td>
</tr>
<tr>
<td>-1</td>
<td>-3</td>
<td>$5\pi/6$</td>
</tr>
</tbody>
</table>

\[
\frac{1|x|^2}{1 \times 1} = \frac{(\xi, \xi)}{(\eta, \eta)}
\]
Out of this table we get

Lemma V.28 Assume \( \alpha, \beta \) are not proportional roots and \( \langle \alpha, \beta \rangle > 0 \), then \( \beta - \alpha \in \Sigma' \). If \( \langle \alpha, \beta \rangle < 0 \) then \( \beta + \alpha \in \Sigma' \).

Next we turn to the study of basis of a root system.

Def. V.29 A basis of \( \Sigma \) is a subset \( \Delta \subset \Sigma \) such that

1. \( \Delta \) is a basis of \( E \)

2. for every \( \beta \in \Sigma' \) all coordinates of \( \beta \) in the basis \( \Delta \) are integers, and all of the same sign.

The elements of \( \Delta \) are called simple roots; the roots \( \Sigma' \) with positive coordinates
are called positive, and the others negative.

Let $\Sigma^+$ be the set of positive roots, then

$$\Sigma = \Sigma^+ \cup (\pm \Sigma^+).$$

Notation: $\Sigma = -\Sigma^\perp$.

**Lemma 5.30** For every $\alpha \neq \beta \in \Delta$,

$$\langle \alpha, \beta \rangle \leq 0.$$

*Proof:* Otherwise $\langle \alpha, \beta \rangle > 0$ and by 5.28:

$$\gamma = \alpha - \beta \in \Sigma^\perp,$$

which contradicts property (2) of a basis.

Let for every $\gamma \in E \setminus \{0\}$,

$$P_\gamma = \{ x \in E : \langle x, \gamma \rangle = 0 \}.$$

Then $P_\gamma$ is a hyperplane and disconnect $E$ into 2 components.

**Def 5.31** (1) A Weyl chamber is a connected component of $E \setminus (\bigcup_{\alpha \in \Sigma} P_\alpha)$.

(2) $\gamma \in E$ is regular if $\langle x, \gamma \rangle \neq 0$

for all $\alpha \in \Sigma^\perp$; that is, $\gamma$ belongs to some Weyl chamber.
Given $\gamma \in E$ regular, let

$$\Sigma^+ (\gamma) = \left\{ \alpha \in \Sigma^* : \langle \alpha, \gamma \rangle > 0 \right\}.$$

Clearly, $\Sigma = \Sigma^+ (\gamma) \cup (-\Sigma^+ (\gamma))$.

Call $\alpha \in \Sigma^+ (\gamma)$ indecomposable if it is not the sum of two element of $\Sigma^+ (\gamma)$. Let

$$\Delta (\gamma) = \left\{ \alpha \in \Sigma^+ (\gamma) : \alpha \text{ is indecomposable} \right\}.$$

Thm IV.32: $\forall \gamma \in E$ regular, $\Delta (\gamma)$ is a basis of $\Sigma^*$ and any basis is of this form.

Proof:

1. Every element in $\Sigma^+ (\gamma)$ is a $\geq 0$ linear combination of elements in $\Delta (\gamma)$.

By contradiction, take $\alpha$ a counterexample with $\langle \alpha, \gamma \rangle$ minimal.
Then \( \alpha \) must be decomposable, that is
\[
\alpha = \beta_1 + \beta_2 \text{ with } \beta_i \in \Sigma^{+} \mathbb{R}.
\]

But then \( \langle \gamma, \mathbf{v} \rangle = \langle \delta, \mathbf{v} \rangle + \langle \sigma, \mathbf{v} \rangle \)
and since \( 0 < \langle \gamma, \mathbf{v} \rangle = \langle \gamma, \mathbf{b} \rangle < \langle \gamma, \mathbf{v} \rangle < \langle \gamma, \mathbf{q} \rangle > 0 \).

Hence both \( \beta_1, \beta_2 \) are a \( \mathbb{Z}_{>0} \)-linear combination of elements in \( \Delta(\mathfrak{g}) \) and \( \gamma \) is \( \lambda \).

(2) If \( \lambda \neq \beta \in \Delta(\mathfrak{g}) \) then \( \langle \alpha, \beta \rangle \leq 0 \).

Otherwise \( \alpha - \beta \) and \( \beta - \alpha \) are roots
hence WLOG \( \lambda - \alpha \in \Sigma^{+} \mathfrak{h} \); but then
\[
\alpha = (\lambda - \beta) + \beta
\]
contradicting the hypothesis that \( \alpha \) is indecomposable.

(3) \( \Delta(\mathfrak{g}) \) is linearly independent.

Otherwise \( \Delta(\mathfrak{g}) = \Delta(\mathfrak{h}) + \mathbb{R} \)

Let \( \sum \lambda_i \mathbf{x} = 0 \) be a linear
\( \lambda \in \Delta(\mathfrak{g}) \).
Define \( \Lambda = \set{\lambda \in \Lambda \mid \lambda > 0} \) and \( \Lambda' = \set{\lambda \in \Lambda' \mid \lambda < 0} \).

Then \( \sum_{\alpha \in \Lambda} \lambda_\alpha = \sum_{\beta \in \Lambda'} (-\lambda_\beta) \leq 0 \).

Thus \( \sum_{\alpha \in \Lambda} \lambda_\alpha < 1 = \sum_{\alpha \in \Lambda} \lambda_\alpha (-\lambda_\beta) < \lambda < 1 \),

\[ \sum_{\alpha \in \Lambda} \lambda_\alpha = 0 = \sum_{\beta \in \Lambda'} (-\lambda_\beta) \leq 0. \]

Which implies
\[ \sum_{\alpha \in \Lambda} \lambda_\alpha \varphi_\alpha = 0 \]

But then \( \sum_{\alpha \in \Lambda} \lambda_\alpha \varphi_\alpha > 0 \),

which implies \( \Lambda = \emptyset \) and \( \Lambda' = \emptyset \).
(4) It follows that \( \Delta \sigma_1 \) is a basis of \( \Sigma \).

(5) Let now \( \Delta \in \Sigma \) be any basis.

Since \( \Delta \) is a basis of \( E \) there exists \( \gamma \in E \) with \( \langle \alpha, \gamma \rangle > 0 \) \( \forall \alpha \in \Delta \) (exercise).

Then clearly \( \gamma \) is regular and:

\[ \Sigma^+ \subset \Sigma^+(\gamma) \quad \text{and} \quad \Sigma^- \subset \Sigma^-(\gamma) \]

Thus we have equality in both cases.

Thus \( \Sigma^+ = \Sigma^+(\gamma) \) and \( \Delta(\gamma) \) consists of indecomposable elements \( \rightarrow \Delta(\gamma) \subset \Delta \)

Hence \( \Delta(\gamma) = \Delta \).

Let now \( \mathcal{W} \) be the subgroup of \( O(E) \)

the orthogonal group of \( E \) generated by the set \( \{ \sigma_\alpha : \alpha \in \Sigma^- \} \) of all reflections.
Then \( w ( \Sigma') = \Sigma' \) for all \( w \in W \), and
\[
W \sigma_{W}^{-1} = \sigma_{W} \sigma_{\sigma (w)}^{-1} \quad \forall w \in W \forall \sigma \in \Sigma'.
\]
As a result, \( W \) permutes the set of Weyl chambers; it also permutes the set of basis of \( \Sigma' \).

Now if \( \mathcal{G} \) is a Weyl chamber and \( \delta , \delta' \in \mathcal{G} \) then \( \Sigma_{\mathcal{G}}^+ \delta = \Sigma_{\mathcal{G}}^+ \delta' \) hence \( \Delta \delta = \Delta \delta' \).

This gives a \( W \)-equivariant bijection between the set of Weyl chambers and the set of basis of \( \Sigma' \).

**Thm 5.32** Let \( \Delta \) be a basis of \( \Sigma' \).

1. \( W \) is generated by \( \{ \sigma_x : x \in \Delta \} \).

2. \( W \) acts simply transitively on the set of basis.

3. \( W \) acts simply transitively on the set of Weyl chambers.
Lemma 5.34: \( \forall \alpha \in \Delta: \alpha \text{ positive } \sum^+ \iff \alpha \) is positive.

Proof: Let \( \beta \in \Sigma^+ \setminus \{\alpha\} \). Then there is \( \gamma \in \Delta \) such that the \( \gamma \)-coordinate in \( \beta \) is \( > 0 \); now \( \sigma_\alpha (\beta) = \beta - (\gamma, \alpha) \alpha \) and the coord. of \( \sigma_\alpha (\beta) \) wrt \( \gamma \) is unchanged.

Hence \( \sigma_\alpha (\beta) \in \Sigma^+ \). \( \square \)

The following is then immediate:

Lemma 5.35: Let \( \sigma = \Sigma \beta \). Then \( \beta \in \Sigma^+ \)

\[ \sigma (\alpha) = \sigma - 2\alpha \quad \forall \alpha \in \Delta. \]

Lemma 5.36: Let \( \alpha_1, \ldots, \alpha_n \in \Delta \), and write \( \sigma_i = \sigma_\alpha \). If \( \sigma_1 - \sigma_n \alpha_n \alpha_n \in \Sigma^+ \), then there is \( 1 \leq i < n \) with

\[ \sigma_i - \sigma_n = \sigma_1 - \sigma_i - \ldots - \sigma_{i-1}. \]
Proof: We write the hypothesis as

$$\sigma_n \cdot \sigma_{n-1}(\alpha_n) \in \Sigma^-,$$

since $$\sigma_n(\alpha_n) = -\alpha_n.$$

Now consider the sequence

$$\sigma_{n-1}(\alpha_n)$$
$$\sigma_{n-2}(\sigma_{n-1}(\alpha_n))$$
$$\vdots$$
$$\sigma_1 \cdot \sigma_{n-1}(\alpha_n) \in \Sigma^-.$$

If $$\sigma_{n-1}(\alpha_n) \in \Sigma^-$$ then we must have

$$\lambda = \lambda_n$$

and hence $$\sigma_{n-1} \cdots \sigma_n = \sigma_{n-1} \cdot \cdots \cdot \sigma_{n-2}.$$ Other-\wedge\hbox{wise} $$\sigma_{n-1}(\alpha_n) \in \Sigma^+$$; consider

then there is $$i \leq n-2$$ minimal such that

$$\sigma_{i+1} \cdots \sigma_{n-1}(\alpha_n) \in \Sigma^+$$

$$\sigma_i \sigma_{i+1} \cdots \sigma_{n-1}(\alpha_n) \in \Sigma^-.$$ Then we must have

$$\sigma_{i+1} \cdots \sigma_{n-1}(\alpha_n) = \alpha_i.$$
\[-\sqrt{5} - 7\]

Hence, \( \sigma_i = \sigma_1 = \sigma_{W(n)} = W \sigma_n W^{-1} = W \sigma_i W \)

that is:

\[ W \sigma_n = \sigma_i W \]

or

\[ \sigma_{i+1} \cdots \sigma_1 \sigma_n = \sigma_i \sigma_{i+1} \cdots \sigma_{n-1}. \]

Multiplying on both sides by \( \sigma_i \cdots \sigma_1 \): we get

\[ \sigma_i \cdots \sigma_1 \sigma_n = \sigma_i \cdots \sigma_{i+1} \sigma_{i+1} \cdots \sigma_{n-1}. \]

We get immediately:

**Lemma IV.37.** Let \( \sigma = \sigma_1 \cdots \sigma_n \)

where \( n \) is minimal such that \( \sigma \) is the number of factors in a product of simple reflections. Then:

\[ \sigma (\sigma_i \sigma_{i+1} \cdots \sigma_{n-1}) = \sigma_i. \]
Proof of Thm V. 33.

Let $W'$ be the subgroup of $W$ generated by $\{ \sigma_\alpha : \alpha \in \Delta \}$.

1. Let $\gamma \in E$ be a regular element. We show that $\exists \sigma \in W'$ such that
\[
\langle \sigma (\gamma), \alpha \rangle > 0 \quad \forall \alpha \in \Delta.
\]
This will imply that if $G$ is the Weyl chamber containing $\gamma$, $\sigma (G')$ is the Weyl chamber associated to the basis $\Delta$.

To this end let $\delta := \sum_\alpha \alpha \in \mathbf{R}^+$ and choose $\sigma^* \in W'$ with $\langle \sigma^* (\gamma), \delta \rangle$ is maximal. Then $\forall \alpha \in \Delta$:
\[
\langle \sigma^* (\gamma), \delta \rangle = \langle \sigma^* \sigma_\alpha (\gamma), \delta \rangle = \langle \sigma \gamma, \alpha \delta \rangle = \langle \sigma \gamma, \delta - \alpha \delta \rangle
\]
which implies $\langle \sigma \gamma, \alpha \rangle > 0 \quad \forall \alpha \in \Delta$.
\[ -\nabla - \delta^2 - \]

and hence \( \langle \sigma(\alpha), \alpha \rangle > 0 \) \( \forall \alpha \in \Delta \) since \( \sigma(\alpha) \) is regular as well. This shows that \( W' \) acts transitively on Weyl chambers hence on \( \Lambda^1 \).

(2) Given \( \alpha \in \Sigma \) \( \exists \in W' \) with \( \sigma(\alpha) \).

Since \( W' \) acts transitively on the set of bases this amounts to show that \( \alpha \in \Sigma \) is element of some basis. But let now \( \gamma \in P_\alpha \setminus (U P_\beta) \) and pick \( \beta \in \Sigma \)

\( \gamma' \) close to \( \gamma \) with \( \langle \gamma', \alpha \rangle = \varepsilon > 0 \) and

\[ \langle \gamma', \beta \rangle > \varepsilon \quad \forall \beta \in \Sigma. \]

Then \( \alpha \in \Sigma^{+|\beta|} \) and \( \alpha \) is indecomposable.

(3) \( \tilde{W} = W' \).

By (2) \( \exists \sigma \in W' \) with \( \sigma(\alpha) \in \Delta \) that is \( \sigma \alpha = \sigma \sigma(\alpha) \sigma' = \in W' \).
(4) $W$ acts freely on the set of basis.

Let $x(\Delta) = \Delta$. Let $\sigma = \sigma_1 \cdots \sigma_n$
minimal product of simple reflections.

Then $\sigma(x_n) \in \Delta \subseteq \Sigma^+$ a contradiction.