A derivation \( D \) of \( \mathfrak{g} \) is an endomorphism \( D : \mathfrak{g} \to \mathfrak{g} \) such that:

\[
D([x, y]) = [Dx, y] + [x, Dy]
\]

for \( x, y \in \mathfrak{g} \).

The vector space of derivations of \( \mathfrak{g} \) is denoted \( \text{Der}(\mathfrak{g}) \); it is a Lie subalgebra of \( \text{gl}(\mathfrak{g}) \).

**Def. 15.1.** The Killing form of a Lie algebra \( \mathfrak{g} \) is the bilinear symmetric form given by \( B : \mathfrak{g} \times \mathfrak{g} \to k \):

\[
(x, y) \mapsto \text{tr}(\text{ad}x \cdot \text{ad}y).
\]

This symmetric form has the following fundamental properties.
Lemma II.2.

(2) \( \forall D \in \text{Der}(g) \) we have:

\[
B_g(Dx, y) = -B_g(x, Dy) \quad \forall x, y \in g
\]

in particular, this holds for \( D = \text{ad}(z) \) \( \forall z \in g \).

(2) \( \forall x \in \text{Aut}(g) \) we have:

\[
B_g(x(x), x(y)) = B_g(x(y), x(x)) \quad \forall x, y \in g.
\]

Proof:

(2) Let's compute

\[
\text{ad}(Dx) = D(x) = [Dx, [x, z]]
\]

\[
= -[x, D([x, z])] + D([x, [x, z]])
\]

\[
= -[x, [Dx, z]] - [x, [x, Dz]] + D([x, [x, z]])
\]
which reads:

\[ \text{ad}(Dx) \text{ad}(Y) = -\text{ad}(X) \text{ad}(DY) \]

\[ - \text{ad}(X) \text{ad}(Y_1) \]

\[ + \text{ad}(X) \text{ad}(Y) \]

and the claim follows by taking traces.

(2) Since \( \alpha \) is a Lie Algebra automorphism, we have

\[ \alpha([x,y]) = [\alpha(x), \alpha(y)] \]

which reads

\[ \alpha \circ \text{ad}(x) = \text{ad}(\alpha(x)) \circ \alpha \]

or

\[ \text{ad}(\alpha(x)) = \alpha \text{ad}(x) \alpha^{-1} \].

But then

\[ B_{g}(\alpha(x), \alpha(y)) = \text{tr}(\alpha(x) \text{ad}(y)) \]

\[ = \text{tr}(\alpha \text{ad}(x) \alpha^{-1} \text{ad}(y)) \]

\[ = \text{tr}(\text{ad}(x) \text{ad}(y)) \]

\[ = B_{g}(x, y). \]
Thus the adjoint representation

\[ \text{ad} : g \rightarrow \text{End}(g) \]

is formed of endomorphisms that are antisymmetric with respect to the Killing form. Clearly if \( g \) is abelian, that is \( \text{ad}(g) = 0 \) we learn nothing new since \( \text{ad} g \) identically zero. It turns out that Lie algebras for which \( \text{ad} g \) is non-degenerate admit a rich well understood structure theory.

**Def. III.3** A real Lie algebra is semisimple if its Killing form is non-degenerate.
Later on we will show that a semisimple Lie algebra is a direct sum of simple non-abelian ideals.

III.2. Orthogonal Symmetric Lie Algebra

Recall that if $(\mathcal{G}, \mathbb{K})$ is a riemannian symmetric pair then on $\mathfrak{g} = \text{Lie}(\mathcal{G})$ we have the Cartan involution $(\mathfrak{g} = \mathfrak{h})_0$ whose set of fixed vectors

$$\{ x \in \mathfrak{g} : \mathfrak{h}(x, x) = 0 \}$$

is the Lie algebra $\mathfrak{h}$ of $\mathbb{K}$. In addition, since

$$\mathfrak{g} \xrightarrow{\text{ad}} \mathfrak{g} \text{Lie}(\mathfrak{g})$$

$$\exp^{-1} \downarrow \quad \downarrow \exp$$

$$\mathfrak{g} \xrightarrow{\exp} \mathbb{G} \xrightarrow{\mathfrak{g}} \text{Lie}(\mathbb{G})$$
Since \( \text{ad} \) is the derivative of the Lie group homomorphism \( \text{Ad} \), we have that \( \text{ad}(\mathfrak{g}) = \text{Lie Ad}(\mathfrak{k}) \) and hence \( \text{ad}(\mathfrak{z}) \) is the Lie algebra of a compact subgroup of \( \text{GL}(\mathfrak{g}) \). This leads us to the following definition:

**Def. III. 4** A Lie subalgebra \( \mathfrak{z} < \mathfrak{g} \) is compactly embedded if \( \text{ad}(\mathfrak{z}) < < \mathfrak{gl}(\mathfrak{g}) \) is the Lie algebra of a compact subgroup of \( \text{GL}(\mathfrak{g}) \).

**Def. III. 5** (1) An orthogonal symmetric Lie algebra is a pair \( (\mathfrak{g}, \Theta) \) consisting of a (real) Lie algebra \( \mathfrak{g} \) and an involutory automorphism \( \Theta \) such that the Lie subalgebra
$u := \{ x \in \mathcal{G} : \varpi(x) = x \}$ is compactly embedded. It is part of the definition that $\varpi + \text{id}_y$ and $\varpi^2 - \text{id}_y$.

(2) An OSL $(y, \varpi)$ is effective if $\mathcal{Z}(y) \cap \mathcal{W} = \{0\}$ where $\mathcal{Z}(y)$ denotes the center of $y$.

Given an OSL we have a direct sum decomposition

$$y = \pi + e$$

where $\pi$ is the $+1$-eigenspace and $e$ is the $-1$-eigenspace of $\varpi$. Here are two simple consequences:
Lemma III. 6

(1) the decomposition $y = u + e$

above is orthonormal wrt the Killing form $B_y$.

(2) Assume $(y, \Theta)$ effective. Then $B_y$ is negative definite on $\mathfrak{m}$.

Proof:

(1) Let $x \in \mathfrak{u}, \Theta \in \mathfrak{e}$. Since $\Theta$

preserves $B_y$

$$B_y(x, y) = B_y(\theta(x), \theta(y)) = B_y(x, -y)$$

$$= -B_y(x, y).$$

(2) By hypothesis ad $(\Theta) = \text{Lie} \mathbf{u}$

where $\mathbf{u} = \mathbf{c} \mathbf{h} (\mathfrak{g})$ is a compact

subgroup. Let $<, >$ be an $\mathbf{u}$-invariant
scalar product on $g$. Then we have

$$\forall x \in L_i^2(u): \quad \langle Zu, u \rangle + \langle u, Zu \rangle = 0 \quad \forall u, v \in g$$

In particular this holds for $ad(x_1)$, $x_1 \in \mathfrak{g}$. Let then $e_1, \ldots, e_n$ be a basis of $g$. We have $\forall x \in \mathfrak{g}$:

$$B_g(x, x) = tr(ad(x)^2) = \sum_{i=1}^{n} \langle (ad(x))^2 e_i, e_i \rangle = -\sum_{i=1}^{n} \|ad(x) e_i\|^2 \leq 0$$

with equality if $ad(x) = 0$, that is $x \in \mathfrak{z}(g) \cap u = 0$.
The following concepts are then central:

Def. \[ \text{Let } (g, \circ) \text{ be an effective } \\]\[ \text{RSL with decomposition } g = u + e \]\[ \text{and Killing form } B_g. \]

(1) \( (g, \circ) \) is of compact type if 
\[ B_g \text{ is negative definite on the whole of } g. \]

(2) \( (g, \circ) \) is called of non-compact type if 
\[ B_g \] is positive definite.

(3) \( (g, \circ) \) is called of unitary type if 
\[ e \] is an abelian ideal.
Remarks III. 8

(1) In the two first cases $B_y$ is non-degenerate and hence $y$ is semisimple.

(2) The third case is equivalent to the property that $[e, e] = 0$ since $[z, e] = e$. 
