A derivation ~~D~~ of $\mathfrak{g}$ is an endomorphism $D : \mathfrak{g} \to \mathfrak{g}$ such that:

$$D([X,Y]) = [DX, Y] + [X, DY]$$
$$\forall X, Y \in \mathfrak{g}.$$

The vector space of derivations ~~of~~ of $\mathfrak{g}$ is denoted $\mathrm{Der}(\mathfrak{g})$; it is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$.

<u>Def. IV.1.</u> The Killing form of a Lie algebra $\mathfrak{g}$ is the bilinear symmetric form given by

$$B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \longrightarrow k$$
$$(X, Y) \longmapsto \mathrm{tr}(\mathrm{ad}\, X \cdot \mathrm{ad}\, Y).$$

This symmetric form has then the following fundamental properties.

Lemma II.2.

(1) $\forall D \in \mathrm{Der}(\mathfrak{g})$ we have:

$$B_{\mathfrak{g}}(DX, Y) = -B_{\mathfrak{g}}(X, DY) \quad \forall X, Y \in \mathfrak{g}$$

in particular this holds for $D = \mathrm{ad}(Z)$ $\forall Z \in \mathfrak{g}$.

(2) $\forall \alpha \in \mathrm{Aut}(\mathfrak{g})$ we have:

$$B_{\mathfrak{g}}(\alpha(X), \alpha(Y)) = B_{\mathfrak{g}}(X, Y) \quad \forall X, Y \in \mathfrak{g}.$$

Proof:

(1) Let's compute

$$\mathrm{ad}(DX)\,\mathrm{ad}(Y)(Z) = [DX, [Y, Z]]$$

$$= -[X, D([Y,Z])] + D([X, [Y, Z]])$$

$$= -[X, [DY, Z]] - [X, [Y, DZ]] + D([X, [Y, Z]])$$

which reads:

$$\mathrm{ad}(DX)\,\mathrm{ad}(Y) = -\,\mathrm{ad}(X)\,\mathrm{ad}(DY)$$
$$-\,\mathrm{ad}(X)\,\mathrm{ad}(Y)\,D$$
$$+\,D\,\mathrm{ad}(X)\,\mathrm{ad}(Y)$$

and the claim follows by taking traces.

(2) Since α is a Lie Algebra automorphism we have $\alpha([X,Y]) = [\alpha(X), \alpha(Y)]$

which reads

$$\alpha \circ \mathrm{ad}(X) = \mathrm{ad}(\alpha(X)) \circ \alpha$$

or $$\mathrm{ad}(\alpha(X)) = \alpha\,\mathrm{ad}(X)\,\alpha^{-1}.$$

But then

$$B_g(\alpha(X), \alpha(Y)) = \mathrm{tr}\left(\mathrm{ad}(\alpha(X))\,\mathrm{ad}\,\alpha(Y)\right)$$
$$= \mathrm{tr}\left(\alpha\,\mathrm{ad}(X)\,\mathrm{ad}(Y)\,\alpha^{-1}\right)$$
$$= \mathrm{tr}\left(\mathrm{ad}(X)\,\mathrm{ad}(Y)\right)$$
$$= B_g(X,Y). \quad \boxed{\text{S}}$$

Thus the adjoint representation

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$

is formed of endomorphisms that are antisymmetric with respect to the Killing form. Clearly if $\mathfrak{g}$ is abelian, that is $\mathrm{ad}(\mathfrak{g}) = 0$ we learn nothing new since $B_{\mathfrak{g}}$ is identically zero. It turns out that Lie algebras for which $B_{\mathfrak{g}}$ is non-degenerate admit a rich well understood structure theory.

<u>Def. III.3</u> A real Lie algebra is semisimple if its Killing form is non-degenerate.

Later on we will show that a semisimple Lie algebra is a direct sum of simple non-abelian ideals.

III.2. Orthogonal Symmetric Lie Algebras

Recall that if (G, K) is a riemannian symmetric pair then on $\mathfrak{g} = \mathrm{Lie}(G)$ we have the Cartan involution $\Theta = D_e\sigma$ whose set of fixed vectors

$$\{ X \in \mathfrak{g} : \Theta(X) = X \}$$

is the Lie algebra $\mathfrak{k}$ of K. In addition, since

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[d, "\exp"] \\ G \arrow[r, "\mathrm{Ad}"'] & GL(\mathfrak{g}) \end{tikzcd}$$

Since ad is the derivative of the Lie group homomorphism Ad, we have that $ad(\mathfrak{k}) = \text{Lie } Ad(K)$ and hence $ad(\mathfrak{k})$ is the Lie algebra of a compact subgroup of $GL(\mathfrak{g})$.

This leads us to the following definitions:

<u>Def. III. 4</u> A Lie subalgebra $\mathfrak{k} < \mathfrak{g}$ is compactly embedded if $ad(\mathfrak{k}) < \mathfrak{gl}(\mathfrak{g})$ is the Lie algebra of a compact subgroup of $GL(\mathfrak{g})$.

<u>Def. III. 5</u> (1) An orthogonal symmetric Lie Algebra (OSL) is a pair $(\mathfrak{g}, \Theta)$ consisting of a (real) Lie algebra $\mathfrak{g}$ and an involutory automorphism Θ such that the Lie subalgebra

$$\mathfrak{u} := \{ X \in \mathfrak{g} : \Theta(X) = X \}$$ is

compactly embedded. It is part of the definition that $\Theta \neq \mathrm{id}_{\mathfrak{g}}$ and $\Theta^2 = \mathrm{id}_{\mathfrak{g}}$.

(2) An OSL $(\mathfrak{g}, \Theta)$ is effective if $Z(\mathfrak{g}) \cap \mathfrak{u} = (0)$ where $Z(\mathfrak{g})$ denotes the center of $\mathfrak{g}$.

Given an OSL we have a direct sum decomposition

$$\mathfrak{g} = \mathfrak{u} + \mathfrak{e}$$

where $\mathfrak{u}$ is the $+1$-eigenspace and $\mathfrak{e}$ is the -1-eigenspace of Θ. Here are two simple consequences:

Lemma III. 6

(1) the decomposition $\mathfrak{g} = \mathfrak{u} + \mathfrak{e}$ above is orthogonal wrt the Killing form $B_{\mathfrak{g}}$.

(2) Assume $(\mathfrak{g}, \theta)$ effective. Then $B_{\mathfrak{g}}$ is negative definite on $\mathfrak{u}$.

Proof:

(1) Let $X \in \mathfrak{u}$, $Y \in \mathfrak{e}$. Since θ preserves $B_{\mathfrak{g}}$:

$$B_{\mathfrak{g}}(X, Y) = B_{\mathfrak{g}}(\theta(X), \theta(Y)) = B_{\mathfrak{g}}(X, -Y)$$
$$= -B_{\mathfrak{g}}(X, Y).$$

(2) By hypothesis $\mathrm{ad}(\mathfrak{u}) = \mathrm{Lie}\, U$ where $U < GL(\mathfrak{g})$ is a compact subgroup. Let $\langle \, , \, \rangle$ be an U-invariant

scalar product on $\mathfrak{g}$. Then we have $\forall\, Z \in \mathrm{Lie}(U)$:

$$\langle Zu, v\rangle + \langle u, Zv\rangle = 0 \quad \forall u, v \in \mathfrak{g}.$$

In particular this holds for $\mathrm{ad}(X)$, $X \in \mathfrak{k}$. Let then $e_1, \dots, e_n$ be a basis of $\mathfrak{g}$. We have $\forall\, X \in \mathfrak{k}$:

$$B_{\mathfrak{g}}(X, X) = \mathrm{tr}(\mathrm{ad}(X)^2)$$

$$= \sum_{i=1}^{n} \langle (\mathrm{ad}(X))^2 e_i, e_i\rangle$$

$$= -\sum_{i=1}^{n} \|\mathrm{ad}(X) e_i\|^2$$

$$\leq 0$$

with equality iff $\mathrm{ad}(X) = 0$, that is $X \in Z(\mathfrak{g}) \cap \mathfrak{k} = (0)$ $\blacksquare$

The following concepts are then central:

<u>Def. III.7</u> Let $(\mathfrak{g}, \Theta)$ be an effective OSL with decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{e}$ and Killing form $B_{\mathfrak{g}}$.

(1) $(\mathfrak{g}, \Theta)$ is of compact type if $B_{\mathfrak{g}}$ is negative definite on the whole of $\mathfrak{g}$.

(2) $(\mathfrak{g}, \Theta)$ is called of non-compact type if $B_{\mathfrak{g}}|_{\mathfrak{e}}$ is positive definite.

(3) $(\mathfrak{g}, \Theta)$ is called of euclidean type if $\mathfrak{e}$ is an abelian ideal.

Remark III.8

(1) In the two first cases $B_{\mathfrak{g}}$ is non-degenerate and hence $\mathfrak{g}$ is semisimple.
(2) the third case is equivalent to the property that $[\mathfrak{e},\mathfrak{e}]=0$ since $[\mathfrak{z},\mathfrak{e}]\subset\mathfrak{e}$.