Remark II. 48 :

Let $\mathfrak{n} \subset \mathfrak{p}$ be a Lie triple system and $N = \operatorname{Exp} \mathfrak{n}$ the corresponding totally geodesic submanifold through o. Let $\mathfrak{g}' := \mathfrak{n} + [\mathfrak{n}, \mathfrak{n}]$ the corresponding Lie subalgebra of $\mathfrak{g}$ and $G' < G$ the corresponding (connected) Lie subgroup. Let $K' = G' \cap K$. Then, if $\Theta = D_e \sigma$ is the Cartan involution we have since $\mathfrak{n} \subset \mathfrak{p}$ that $\Theta(\mathfrak{n}) = -\mathfrak{n} = \mathfrak{n}$ and $\Theta([\mathfrak{n}, \mathfrak{n}]) = [\Theta(\mathfrak{n}), \Theta(\mathfrak{n})] = [\mathfrak{n}, \mathfrak{n}]$.

Hence $\Theta(\mathfrak{g}') = \mathfrak{g}'$ and since G' is connected $\sigma(G') = G'$. Set $\sigma' := \sigma|_{G'}$.

Then we have $K' := K \cap G' \subset (G^\sigma) \cap G'$
$= (G')^{\sigma'}$

And $K' = K \cap G' \supset (G^\sigma)^\circ \cap G' \supset ((G')^{\sigma'})^\circ$

the last inclusion following from the fact that $(G^\sigma)^\circ \cap G'$ is an open subgroup of $(G')^{\sigma'}$ and hence must contain $((G')^{\sigma'})^\circ$.

Thus (G', K') is a riemannian symmetric pair with associated symmetric space N.

II.7. Examples.

II.7.1. Riemannian symmetric pair $(SL(n,\mathbb{R}), SO(n))$.

We saw in Example II.30.1 that $\sigma(g) := {}^t g^{-1}$ is an involution on $SL(n,\mathbb{R}) = G$ with fixed point group $K = SO(n)$. Since $SO(n)$ itself is compact, $(SL(n,\mathbb{R}), SO(n))$ is a RSP. Recall that the Lie group exponential for ~~$SL(n,\mathbb{R})$~~ $GL(n,\mathbb{R})$ is given by $\exp : M_{n,n}(\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$,

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

From which follows:

$$^t(\exp X) = \exp {}^t X$$

and hence

$$\sigma(\exp tX) = \exp(-t\,{}^t X)$$

which implies that the Cartan involution Θ is given by $\Theta(X) = -{}^t X$.

Let $\mathfrak{sl}(n,\mathbb{R}) = \{ X \in M_{n,n}(\mathbb{R}) : \operatorname{tr} X = 0 \}$ be the Lie Algebra of $SL(n,\mathbb{R})$. Then

$$\mathfrak{k} = \{ X \in \mathfrak{sl}(n,\mathbb{R}) : X + {}^t X = 0 \}$$

$$\mathfrak{p} = \{ X \in \mathfrak{sl}(n,\mathbb{R}) : X = {}^t X \}.$$

Thus $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is just the decomposition of a matrix as sum of a symmetric and antisymmetric one.

We have seen that G-invariant riemannian

metrics on $M = G/K$ are in one-one correspondence with $Ad(K)$-invariant scalar products on $\mathfrak{p}$. Recall that

$$Ad : SL(n,\mathbb{R}) \longrightarrow GL(\mathfrak{sl}(n,\mathbb{R}))$$

is given by $Ad(g)(X) = g X g^{-1}$.

To obtain an example of invariant scalar product on $\mathfrak{p}$ recall that

$$M_{n,n}(\mathbb{R}) \times M_{n,n}(\mathbb{R}) \longrightarrow \mathbb{R}$$
$$(A,B) \longmapsto tr({}^tA \cdot B)$$

is a scalar product on $M_{n,n}(\mathbb{R})$, in fact it corresponds to the standard scalar product under the identification of $M_{n,n}(\mathbb{R})$ with $\mathbb{R}^{n^2}$ and it is clearly invariant under $O(n)$-conjugation. Its restriction to $\mathfrak{p}$ is then given by $\mathfrak{p} \times \mathfrak{p} \to \mathbb{R}$

$$(A,B) \longmapsto tr(A \cdot B).$$

As model for $M = SL(n,\mathbb{R})/SO(n)$ we can take

$$\mathcal{P}^1(n) = \{ S \in M_{n,n}(\mathbb{R}) : {}^tS = S,\ \det S = 1,\ S \gg 0 \}$$

with $SL(n,\mathbb{R})$ action given by

$$g_* S = g S {}^t g .$$

By Thm II.39 the riemannian exponential map $\mathrm{Exp} : M \longrightarrow \mathcal{P}^1(n)$ is given by

$$\begin{aligned} \mathrm{Exp}\, X &= \exp(X)_* I \\ &= \exp(X)\, {}^t\exp(X) \\ &= \exp(2X). \end{aligned}$$

Let $\mathfrak{a} = \left\{ \operatorname{diag}(x_1, \dots, x_n) : \sum_{i=1}^{n} x_i = 0 \right\}$

Then clearly $\mathfrak{a} \subset \mathfrak{p}$ and $[\mathfrak{a}, \mathfrak{a}] = 0$ which implies that $\mathfrak{a}$ is a Lie triple system and hence $F := \operatorname{Exp}(\mathfrak{a})$ is a totally geodesic subspace of $\mathcal{P}^1(n)$. Let then $X_1, X_2 \in \mathfrak{a}$, what is the distance $d(\operatorname{Exp} X_1, \operatorname{Exp} X_2)$? So $\exp X_2 \in SL(n, \mathbb{R})$, and by invariance of distance we have:

$$d(\operatorname{Exp} X_1, \operatorname{Exp} X_2) = d\Big(\underbrace{\exp\left(-\tfrac{X_2}{2}\right)_* \operatorname{Exp} X_1}_{\operatorname{Exp}(X_1 - X_2)}, \operatorname{Id}\Big)$$

$$= \| X_1 - X_2 \|$$

Thus: $\operatorname{Exp} : \mathfrak{a} \longrightarrow F$ is an isometry between the Euclidean space $\mathfrak{a}$ and the totally geodesic subspace $F \subset \mathcal{P}^1(n)$.

Such a totally geodesic subspace is called flat. Observe $\dim F = n-1$.

III. Decomposition of symmetric spaces.

The main objective of this chapter is to show that every simply connected riemannian symmetric space is a riemannian product of three types of symmetric spaces
- euclidean space
- symmetric space of compact type
- symmetric space of non-compact type.

So far our study of the geometry of symmetric spaces relied upon their relationship with riemannian symmetric pairs. In order to prove the decomposition

theorem we will relate riemannian symmetric pairs to a third class of objects namely orthogonal symmetric Lie algebras. We will first prove a decomposition thm for orthogonal symmetric Lie algebras and then "integrate" it back on the level of ~~Lie groups~~ riemannian symmetric pairs and associated symmetric spaces.

In the course of the development we will meet a central concept in Lie theory, namely the one of semisimple Lie algebra.

III. 1. The Killing form.

Let $\mathfrak{g}$ be a Lie algebra (over an arbitrary field). As usual $\mathfrak{gl}(\mathfrak{g})$ denotes the space of endomorphisms of $\mathfrak{g}$ seen as a Lie algebra, namely for $A, B \in \mathfrak{gl}(\mathfrak{g})$: $[A, B] = AB - BA$.

Recall that
$$\begin{aligned} \mathrm{ad}: \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ X &\longmapsto \{Y \to [X, Y]\} \end{aligned}$$

is the adjoint representation of the Lie algebra $\mathfrak{g}$ and a Lie Algebra homomorphism. In addition, $\mathrm{ad}(X)$ is an example of an endomorphism of $\mathfrak{g}$ called derivation.