Remark III.8

(1) In the two first cases $B_{\mathfrak{g}}$ is non-degenerate and hence $\mathfrak{g}$ is semisimple.

(2) the third case is equivalent to the property that $[e, e] = 0$ since $[\mathfrak{z}, e] \subset e$.

Now we move to the decomposition theorem for OSL's.

Thm III.9 Let $(\mathfrak{g}, \Theta)$ be an effective OSL-algebra. Then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

is a direct sum of Θ-stable ideals such that :

(1) the decomposition is orthogonal wrt the Killing form of $\mathfrak{g}$.

(2) $(\mathfrak{g}_0, \theta|_{\mathfrak{g}_0})$, $(\mathfrak{g}_+, \theta|_{\mathfrak{g}_+})$, $(\mathfrak{g}_-, \theta|_{\mathfrak{g}_-})$ are respectively OSL's of euclidean, compact and non-compact type.

We begin with a very basic construction. Let thus $\mathfrak{g} = \mathfrak{k} + \mathfrak{e}$ be the standard decomposition and let $\mathcal{U} < GL(\mathfrak{g})$ be the compact connected subgroup with

$$\operatorname{Lie} \mathcal{U} = \operatorname{ad}_{\mathfrak{g}}(\mathfrak{k}).$$

Observe that since $\operatorname{ad}_{\mathfrak{g}}(X)(\mathfrak{e}) \subset \mathfrak{e}$ we deduce that the subspace $\mathfrak{e}$ is $\mathcal{U}$-invariant. Observe also (exercise) that $\mathcal{U} < \operatorname{Aut} \mathfrak{g}$.

Let $\langle\, , \rangle$ be an $\mathcal{U}$ – invariant scalar product on $\mathfrak{e}$. Then there is $A \in \mathrm{End}(\mathfrak{e})$ uniquely determined by

$$B_{\mathfrak{g}}(x,y) = \langle Ax, y\rangle \quad \forall x,y \in \mathfrak{e}.$$

Since $B_{\mathfrak{g}}$ is symmetric, A is symmetric wrt $\langle\, , \rangle$.

Since $\mathcal{U} < \mathrm{Aut}\,\mathfrak{g}$ it follows from lemma IV.2(2) that $B_{\mathfrak{g}}$ is $\mathfrak{u}$ – invariant which reads:

$$\langle A k(x), k(y) \rangle = \langle Ax, y\rangle \quad \forall x,y \in \mathfrak{e}$$
$$\forall k \in \mathcal{U}$$

which implies:

$$A \cdot k = k \cdot A \quad \forall k \in \mathcal{U}.$$

and hence

$$A \cdot \mathrm{ad}_{\mathfrak{g}}(u) = \mathrm{ad}_{\mathfrak{g}}(u) \cdot A \quad \forall u \in \mathfrak{u}.$$

Let $f_1, \dots, f_n$ be an ONB consisting of eigenvectors of A with corresponding eigenvalues $\beta_1, \dots, \beta_n$. Define

$$e_0 := \sum_{\beta_i = 0} \mathbb{R} f_i$$

$$e_+ := \sum_{\beta_i > 0} \mathbb{R} f_i$$

$$e_- := \sum_{\beta_i < 0} \mathbb{R} f_i .$$

Since V and $ad_g(x)$ commutes with A, the eigenspaces of A are V and $ad_g(x)$ invariant and so are e_0, e_+, e_-.

Lemma III. 10.

(1) $\mathfrak{e}_0$ is the null space of $B_{\mathfrak{g}}$ in $\mathfrak{g}$

that is $= \{ X \in \mathfrak{g} : B_{\mathfrak{g}}(X, Y) = 0 \ \forall Y \in \mathfrak{g} \}$.

(2) $[\mathfrak{e}_0, \mathfrak{e}] = 0$ and $\mathfrak{e}_0$ is an abelian ideal in $\mathfrak{g}$.

(3) $[\mathfrak{e}_-, \mathfrak{e}_+] = 0$.

Proof:

(1). Let $\mathfrak{g}^\perp := \{ X \in \mathfrak{g} : B_{\mathfrak{g}}(X, Y) = 0$
$\forall Y \in \mathfrak{g} \}$.

Since $B_{\mathfrak{g}}$ is Θ-invariant so is $\mathfrak{g}^\perp$.

Hence $\mathfrak{g}^\perp = (\mathfrak{g}^\perp \cap \mathfrak{u}) \oplus (\mathfrak{g}^\perp \cap \mathfrak{e})$.

Since $(\mathfrak{g}, \Theta)$ is effective, by lemma IV. 6. (2)

$B_{\mathfrak{g}}|_{\mathfrak{u} \times \mathfrak{u}}$ is negative definite, hence

$\mathfrak{g}^{\perp} \cap \mathfrak{u} = (0)$ and $\mathfrak{g}^{\perp} \subset \mathfrak{e}$. Thus

$$\mathfrak{g}^{\perp} = \{ x \in \mathfrak{e} : \langle A x, y \rangle = 0 \ \forall y \in \mathfrak{e} \}$$

$$= \operatorname{Ker} A = \mathfrak{e}_0 .$$

$\square$

(2) $[\mathfrak{e}_0, \mathfrak{e}] \subset \mathfrak{u}$. But now if

$x \in \mathfrak{e}_0,\ y \in \mathfrak{e},\ z \in \mathfrak{u}$:

$$B_{\mathfrak{g}}([x, y], z) = -B([y, x], z)$$

$$= B(x, [y, z])$$

$$= \langle A x, \underbrace{[y, z]}_{\mathfrak{e}} \rangle = 0$$

Since $B_{\mathfrak{g}}|_{\mathfrak{u} \times \mathfrak{u}} \ll 0$ this implies

$[x, y] = 0$ hence $[\mathfrak{e}_0, \mathfrak{e}] = 0$.

In particular $[\mathfrak{e}_0, \mathfrak{e}_0] = 0$ and

$$[\mathfrak{e}_0, \mathfrak{g}] = \underbrace{[\mathfrak{e}_0, \mathfrak{e}]}_{0} + \underbrace{[\mathfrak{e}_0, \mathfrak{u}]}_{\subset \mathfrak{e}_0} \subset \mathfrak{e}_0 .$$

(3) Let $X \in e_-$, $Y \in e_+$, $Z \in \mathfrak{u}$:

$$B([X, Y], Z) = -B(Y, [X, Z])$$

but $Y \in e_-$, $[X, Z] \in e_+$ hence

$$-B(Y, [X, Z]) = -\langle \underbrace{AY}_{e_-}, \underbrace{[X, Z]}_{e_+} \rangle = 0$$

which implies $[X, Y] = 0$.

$\boxed{\xi}$

It is now natural to define:

$$\mathfrak{u}_+ := [e_+, e_+], \quad \mathfrak{u}_- := [e_-, e_-].$$

Lemma IV.11: $\mathfrak{u}_+$ and $\mathfrak{u}_-$ are orthogonal wrt $B_{\mathfrak{g}}$.

Proof: let $X_\pm, Y_\pm$ be in $e_\pm$.

$$B_{\mathfrak{g}}\left([X_+,Y_+],[X_-,Y_-]\right)$$

$$= - B_{\mathfrak{g}}\left(Y_+,[X_+,[X_-,Y_-]]\right).$$

Jacobi gives:

$$[X_+,[X_-,Y_-]] + [Y_-,\underbrace{[X_+,X_-]}^{0 \text{ by II.10(3)}}] + [X_-,\overbrace{[Y_-,X_+]}^{0}]$$

$$= 0$$

$\boxed{}$

Finally we define $\mathfrak{u}_0$ as the orthogonal complement of $\mathfrak{u}_+ \oplus \mathfrak{u}_-$ in $\mathfrak{u}$ wrt $B_{\mathfrak{g}}$.

Lemma III. 12.

(1) $\mathfrak{u}_0, \mathfrak{u}_+, \mathfrak{u}_-$ are ideals in $\mathfrak{u}$.

(2) $[\mathfrak{u}_0, e_-] = [\mathfrak{u}_0, e_+] = 0$

(3) $[\mathfrak{u}_-, e_0] = [\mathfrak{u}_-, e_+] = 0$

(4) $[\mathfrak{u}_+, e_0] = [\mathfrak{u}_+, e_-] = 0$.

Proof:

(1) recall that $\mathrm{ad}(\mathfrak{u})$ preserves $\mathfrak{e}_+$ and $\mathfrak{e}_-$.

Thus if $Z \in \mathfrak{u}$, $X, Y \in \mathfrak{e}_+$:

$$\mathrm{ad}(Z)([X,Y]) = [\underbrace{\mathrm{ad}(Z)X}_{\in \mathfrak{e}_+}, \underbrace{Y}_{\in \mathfrak{e}_+}] + [\underbrace{X}_{\in \mathfrak{e}_+}, \underbrace{\mathrm{ad}(Z)Y}_{\in \mathfrak{e}_+}].$$

Which shows that $\mathfrak{u}_+ := [\mathfrak{e}_+, \mathfrak{e}_+]$ is an ideal in $\mathfrak{u}$ and so is $\mathfrak{u}_-$.

If now $Z \in \mathfrak{u}$ and $X \in \mathfrak{u}$ is orthogonal to $\mathfrak{u}_+ \oplus \mathfrak{u}_-$ we have $\forall\, Y \in \mathfrak{u}_+ \oplus \mathfrak{u}_-$:

$$B(\mathrm{ad}(Z)X, Y) = -B(X, \underbrace{\mathrm{ad}(Z)Y}_{\in \mathfrak{u}_+ \oplus \mathfrak{u}_-})$$

$$= 0$$

hence $\mathfrak{u}_0$ is an ideal as well.

(2) $[\mathfrak{u}_0, \mathfrak{e}_-] = 0$: we know that $\mathrm{ad}(\mathfrak{u})$ preserves $\mathfrak{e}_-$, in particular $[\mathfrak{u}_0, \mathfrak{e}_-] \subset \mathfrak{e}_-$.

Now pick $Z \in u_0$, $X, Y \in e_-$:

$$B([Z,X],Y) = B(Z, \underbrace{[X,Y]}_{\in u_-}) = 0$$

since u_0 is orthogonal to u_-. But $B|_{e_- \times e_-}$ is non-degenerate hence $[Z, X] = 0$.

The same argument gives $[u_0, e_+] = 0$.

(3) ~~Let $Z \in u_-$, $Y \in e_0 \pm e_+$:~~

~~$B([Z,X],Y) =$~~

$$[u_\pm, e_0] = [[e_\pm, e_\pm], e_0]$$

$$\underset{\text{Jacobi}}{\subset} [e_\pm, [e_\pm, e_0]] = 0$$

(by lemma III.14.(2))

and $[u_+, e_-] = [[e_+, e_+], e_-]$

$$\underset{\text{Jacobi}}{\subset} [e_+ [e_+, e_-]] = 0$$

(by lemma III.14.(3))

$\boxed{\Sigma}$

Corollary III.13

(1) $\mathfrak{e}_0$ is an abelian ideal in $\mathfrak{g}$.

(2) $\mathfrak{u}_0 \oplus \mathfrak{e}_0$, $\mathfrak{u}_- \oplus \mathfrak{e}_-$, $\mathfrak{u}_+ \oplus \mathfrak{e}_+$
are ideals in $\mathfrak{g}$ and pairwise orthogonal.

Proof:

(1) is lemma III.10.(2)

(2) We know that $\mathfrak{e} = \mathfrak{e}_0 \oplus \mathfrak{e}_- \oplus \mathfrak{e}_+$
and $\mathfrak{u} = \mathfrak{u}_0 \oplus \mathfrak{u}_- \oplus \mathfrak{u}_+$ are orthogonal decompositions. The orthogonality statement follows then from the fact that $\mathfrak{g} = \mathfrak{u} + \mathfrak{e}$ is an orthogonal decomposition.

$\mathfrak{u}_0 \oplus \mathfrak{e}_0$: We know (lemma III.10) that $\mathfrak{e}_0$ is an ideal in $\mathfrak{g}$ and (lemma III.12) that $\mathfrak{u}_0$ is an ideal in $\mathfrak{u}$.

It remains to check what happens with $[u_0, e]$: now $\mathrm{ad}_{\mathfrak{g}}(u)$ preserves the decomposition $e_0 \oplus e_- \oplus e_+$ and hence $[u_0, e_0] \subset e_0$. Furthermore III. 12 (2) gives $[u_0, e_- \oplus e_+] = 0$, which implies $[u_0, e] \subset e_0$ and proves that $u_0 + e_0$ is an ideal in $\mathfrak{g}$.

$u_- \oplus e_-$:

We compute $[u_- + e_-, \mathfrak{g}]$;

$$[u_- + e_-, u] \subset \underbrace{[u_-, u]}_{\substack{\subset u_- \\ (\text{III}.12.(1))}} + \underbrace{[e_-, u]}_{\subset e_-}$$

$$\subset u_- + e_- .$$

$$[u_- + e_-, e] \subset [u_-, e] + [e_-, e]$$

But $[u_-, e] \subset \underbrace{[u_-, e_-]}_{\subset e_-} + \underbrace{[u_-, e_0 + e_+]}_{= 0 \text{ by III}.12(3)}$

Thus: $[\mathfrak{u}_-, e] \subset e_-$.

Finally: $[e_-, e] \subset [e_-, e_-] + [e_-, e_0] + [e_-, e_+]$

By definition $\mathfrak{u}_- = [e_-, e_-]$; and

$[e_-, e_0] = 0$ by III. 10 (2) while $[e_-, e_+] = 0$

by III. 10. (3).

The argument for $\mathfrak{u}_+$ & $\mathfrak{g}_+$ is the same.

$\boxed{\S}$

Now we proceed to the definition of the decomposition as announced in the Theorem III. 9.

Observe that $\mathfrak{g}_\pm = \mathfrak{u}_\pm \oplus e_\pm \neq 0$ iff $e_\pm \neq 0$ in which case $\mathfrak{u}_\pm \neq 0$ since otherwise the Killing form of $\mathfrak{g}_\pm$ would be identically zero.

It can however happen that $e_0 = 0$ but $u_0 \neq 0$ in which case $\mathfrak{g}_0 = u_0$ is not a good definition since the restriction of Θ to $\mathfrak{g}_0$ would be the identity. Thus we proceed as follows by distinguishing cases:

(1) $e_0 \neq 0$: we define

$$\mathfrak{g}_0 = u_0 + e_0, \quad \mathfrak{g}_- = u_- + e_-, \quad \mathfrak{g}_+ = u_+ + e_+$$

and the involution

$$\Theta_\mu := \Theta|_{\mathfrak{g}_\mu}, \quad \mu \in \{0, +, -\}.$$

Then Θ_μ is an involution on $\mathfrak{g}_\mu$.

(2) $e_0 = 0$, $e_- \neq 0$:

$$\mathfrak{g}_0 = 0, \quad \mathfrak{g}_- = u_0 \oplus u_- \oplus e_-$$

$$\mathfrak{g}_+ = u_+ \oplus e_+ .$$

and $\Theta_{\mp}$ are just obtained by restricting Θ to $g_{\mp}$.

(3) $e_0 = 0$, $e_- = 0$

Then $g_0 = 0$, $g_- = 0$ and

$g_+ = u_0 \oplus u_+ \oplus e_+$.

Now we have to verify that g_0, g_-, g_+ have the desired type.

First $g_0 = u_0 + e_0$ is of euclidean type since by lemma IV.10 e_0 is an abelian ideal in g hence in g_0.

We know by definition that B_g is negative definite on e_- and positive definite on e_+. What remains to be seen is that the Killing form of $g_\pm$

coincides with the restriction of the Killing form of $\mathfrak{g}$! This follows from the following general fact:

Lemma III.14:

If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{n} \trianglelefteq \mathfrak{g}$ an ideal we have: $B_{\mathfrak{n}} = B_{\mathfrak{g}}|_{\mathfrak{n}\times\mathfrak{n}}$.

This is left as an exercise in linear algebra.

Finally we also leave as an exercise to verify that in each case $\mathfrak{u}_0, \mathfrak{u}_-, \mathfrak{u}_+$ are compactly embedded in resp. $\mathfrak{g}_0, \mathfrak{g}_-, \mathfrak{g}_+$.

So far we have introduced three categories of objects:

1. Symmetric spaces $M \ni o$
2. Riemannian symmetric pair (G, K)
3. Orthogonal symmetric Lie algebra $(\mathfrak{g}, \Theta)$.

For OSL's we proved a decomposition theorem for effective OSL's. If now (G, K) is a riemannian symmetric pair and $\mathfrak{g} = \mathrm{Lie}\, G$, $\Theta = D_e \sigma$ the associated OSL, then:

$$Z(\mathfrak{g}) \cap \mathfrak{k} = \mathrm{Lie}\,(Z(G) \cap K)$$

and thus $(\mathfrak{g}, \Theta)$ is effective iff $Z(G) \cap K$ is discrete. Accordingly

We define

<u>Def. III. 15</u> A RSP (G,K) is effective

if $Z(G)\cap K$ is discrete.

Now we observe the following simple

<u>Lemma III. 16.</u>

Let $G = Is(M)^{0}$, $K = Stab_{G}(o)$.

If $N \triangleleft G$ is a normal subgroup

of G contained in K, then $N = e$.

In particular (G,K) is effective.

<u>Proof</u>: We have $gKg^{-1} = Stab_{G}(g_{*}o)$

and thus if $N \subset K$ and is normal in G,

we get $N \subset \bigcap_{g\in G} gKg^{-1} = \bigcap_{m\in M} Stab_{G}(m) = (e)$.

Observe now that ~~Z(G)~~ any subgroup

of $Z(G)$ is normal in G and hence

$Z(G)\cap K = (e)$. $\boxed{\text{B}}$

<u>Def. III. 17</u> : An effective RSP (G, K) is said of compact, non-compact, euclidean type if the corresponding OSL is.

In view of lemma III. 16. the following definition then makes sense:

<u>Def. III 18</u> : A Riemannian symm. space M is said of ~~to~~ compact, non-compact, euclidean type if the corresponding RSP $(Is(M)^{\circ}, Stab_{Is(M)^{\circ}}(o))$ is.

With the help of some basic Lie theory we can now globalize Thm III. 9:

Thm III.19. Let M be a simply connected riemannian symmetric space. Then M is the riemannian product

$$M_0 \times M_- \times M_+$$

of symmetric spaces of euclidean, compact and non-compact type.

Proof: Let $G = Is(M)^0$, $K = Stab_G(o)$, $\sigma(g) = s_o g s_o$ and $\mathfrak{g} = \mathrm{Lie}\, G$, $\Theta = D_e\sigma$ the associated OSL. Since $(\mathfrak{g}, \Theta)$ is effective we can apply Thm III.9 and get

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_- \oplus \mathfrak{g}_+$$

the decomposition into types.

Let G_0, G_-, G_+ be the Lie subgroups of G corresponding to $\mathfrak{g}_0$, $\mathfrak{g}_-$, $\mathfrak{g}_+$.

We observe that G_0, G_-, G_+ are normal subgroups of G and that all their pairwise intersections are discrete.

Now observe: ~~$G_\alpha \cap G_\beta \subset$~~

$$[G_\alpha, G_\beta] \subset G_\alpha \cap G_\beta \quad \alpha \neq \beta$$

and since $[G_\alpha, G_\beta]$ is connected (Why?) and $G_\alpha \cap G_\beta$ discrete, we conclude

$$[G_\alpha, G_\beta] = (e) \quad \forall\, \alpha \neq \beta.$$

Thus the map

$$G_0 \times G_- \times G_+ \xrightarrow{\Psi} G$$

$$(x, y, z) \longmapsto x \cdot y \cdot z$$

is a homomorphism which on the level of Lie algebras induces:

$$\mathfrak{g}_0 \times \mathfrak{g}_- \times \mathfrak{g}_+ \xrightarrow{D_e\Psi} \mathfrak{g}$$

$$(X, Y, Z) \longmapsto X + Y + Z$$

an isomorphism. Hence by classical Lie theory we got that the lift of Ψ to the universal coverings:

$$\tilde{\Psi} : \tilde{G}_0 \times \tilde{G}_- \times \tilde{G}_+ \longrightarrow \tilde{G}$$

is a Lie group isomorphism, which at the level of Lie algebras induces $D_e\Psi$.

Let $p : \tilde{G} \longrightarrow G$ be the covering homomorphism. Then

$$\tilde{G}/(p^{-1}(K))^\circ \longrightarrow \tilde{G}/p^{-1}(K) \simeq G/K \simeq M$$

is a connected covering of M and since M is simply connected this implies

$$p^{-1}(K) = (p^{-1}(K))^\circ$$

is connected. Thus $p^{-1}(K) < \tilde{G}$ is the Lie subgroup associated to $\mathfrak{z} < \mathfrak{g}$.

Let then K_0, K_-, K_+ be the Lie subgroups of $\tilde{G}_0, \tilde{G}_-, \tilde{G}_+$ corresponding to $\mathfrak{z}_0 < \mathfrak{g}_0$, $\mathfrak{z}_- < \mathfrak{g}_-$, $\mathfrak{z}_+ < \mathfrak{g}_+$. Then $K_0 \times K_- \times K_+$ is the Lie subgroup of $\tilde{G}_0 \times \tilde{G}_- \times \tilde{G}_+$ corresponding to $\mathfrak{z}_0 \times \mathfrak{z}_- \times \mathfrak{z}_+$ and since $\tilde{\psi}$ is a Lie group isomorphism this implies $\tilde{\psi}(K_0 \times K_- \times K_+) = p^{-1}(K)$ in particular K_0, K_-, K_+ are closed subgroups of $\tilde{G}_0, \tilde{G}_-, \tilde{G}_+$. One verifies that $(\tilde{G}_0, K_0)$, $(\tilde{G}_-, K_-)$ $(\tilde{G}_+, K_+)$ are riemannian symmetric pairs by lifting σ to $\tilde{G}$. Finally the diffeomorphism $\tilde{\psi}$ induces a $\tilde{\psi}$-equivariant diffeom.

$$\tilde{G}_0/K_0 \times \tilde{G}_-/K_- \times \tilde{G}_+/K_+ \longrightarrow \tilde{G}/p^{-1}(K) = M.$$

$\blacksquare$