Remark III.8

(1) In the two first cases $B_g$ is non-degenerate and hence $g$ is semi-simple.
(2) The third case is equivalent to the property that $[e,e] = 0$ since $[\lambda, e]ce.$

Now we move to the decomposition theorem for DSL's.

Thm III.3 Let $(\gamma, \otimes)$ be an effective DSL-algebra. Then

$$g = g_\gamma \oplus g_\gamma \oplus g_\gamma$$

is a direct sum of $\gamma$-stable ideals such that $\cdot$
(1) the decomposition is orthogonal with the Killing form of \( \mathfrak{g} \).

(2) \((\mathfrak{g}_0, \theta | \mathfrak{g}_0), (\mathfrak{g}_+, \theta | \mathfrak{g}_+), (\mathfrak{g}_-, \theta | \mathfrak{g}_-)\) are respectively DSL's of euclidion, compact and non-compact type.

We begin with a very basic construction. Let thus \( \mathfrak{g} = \mathfrak{u} + \mathfrak{c} \) be the standard decomposition and let \( \mathfrak{U} \subset \mathfrak{gl}(\mathfrak{g}) \) be the compact connected subgroup with \( \text{Lie } \mathfrak{U} = \text{ad}_\mathfrak{g}(\mathfrak{u}) \).

Observe that since \( \text{ad}_\mathfrak{g}(\mathfrak{x})(\mathfrak{c}) \in \mathfrak{c} \) we deduce that the subspace \( \mathfrak{c} \) is \( \mathfrak{U} \)-invariant. Observe also (exercise) that \( \mathfrak{U} \subset \text{Aut}\mathfrak{g} \).
Let $<,>$ be an $U$-invariant scalar product on $E$. Then there is $A \in \text{End}(E)$ uniquely determined by

$$B(y)(x,y) = \langle Ax, y \rangle$$

Since $B(y)$ is symmetric, $A$ is symmetric w.r.t $<,>$. Since $U \subset \text{Aut}(y)$ it follows from Lemma 1.2 (2) that $B(y)$ is $U$-invariant which reads:

$$\langle A k(x), k(y) \rangle = \langle Ax, y \rangle \forall x, y \in E$$

Which implies:

$$A \cdot k = k \cdot A \quad \forall k \in U.$$ 

and hence

$$A \cdot \text{ad}_g (u) = \text{ad}_g (u) \cdot A \quad \forall u \in U.$$
Let \( \mathcal{E} \) be an \( \mathbb{R}^n \) consisting of eigenvectors of \( A \) with corresponding eigenvalues \( \beta_1, \ldots, \beta_n \). Define

\[
e_0 = \sum_{\beta_i = 0} R f_i
\]

\[
e_+ = \sum_{\beta_i > 0} R f_i
\]

\[
e_- = \sum_{\beta_i < 0} R f_i
\]

Since \( V \) and \( \text{ad} \gamma(n) \) commutes with \( A \), the eigenspace of \( A \) are \( V \) and \( \text{ad} \gamma(n) \) invariant and so are \( e_0, e_+, e_- \).
Lemma IV. 10.

(1) $e_0$ is the null space of $B_y$ in $y$ that is $\mathcal{N}(B_y) = \{ x \in y : B_y(x, r) = 0 \ \forall r \in y \}$.

(2) $[e_0, e] = 0$ and $e_0$ is an abelian ideal in $y$.

(3) $[e_-, e_+] = 0$.

Proof:

(4). Let $y^+ = \{ x \in y : B_y(x, r) = 0 \ \forall r \in y \}$.

Since $B_y$ is $\Theta$-invariant so is $y^+$.

Hence $y = (y^+ \cap u) \oplus (y^+ \cap e)$.

Since $(y, \Theta)$ is effective, by lemma IV. 6 (2) $B_y|_{u \times u}$ is non-degenerate, hence
\[
g^+ \cap u = (0) \quad \text{and} \quad g^+ \subseteq e. \quad \text{Thus}
\]
\[
y^+ = \{ x \in e : \langle Ax, y \rangle = 0 \quad \forall y \in e \}
\]
\[
= \ker A = e_0.
\]

(2) \([e_0, e] \subseteq u\). But now if \(x \in e_0, y \in e, z \in u\):

\[
B(x, [y, z]) = -B([y, x], z) = B(x, [y, z]) = \langle Ax, [y, z] \rangle = 0
\]

Since \(B\) is skew-symmetric, this implies \([x, y] = 0\) hence \([e_0, e] = 0\).

In particular \([e_0, e_0] = 0\) and

\[
[e, g] = [e_0, e] + [e_0, u] \subseteq e_0.
\]

\[
[0, e_0] \subseteq e_0.
\]
(3) Let $x \in e_-$, $y \in e_{+1}$, $z \in e_+$.

$B([x,y], z) = -B(y, [x,z])$

but $y \in e_-$, $[x,z] \in e_{+1}$ hence

$-B(y, [x,z]) = -\langle Ay, [x,z] \rangle = 0$

which implies $[x,z] = 0$.

It is now natural to define:

$u_+ := [e_+, e_+]$, $u_- := [e_-, e_-]$.

**Lemma 3.11**: $u_+$ and $u_-$ are orthogonal

**Proof**: Let $x_\pm, \Gamma_\pm \in e_\pm$. 
\[ B_g \left( [x_+], [x_-] \right) \]

\[ = - B_g \left( [x_+], [x_-, x_-] \right) \]

Jaccsi gives:

\[ [x_+, [x_-, x_-]] + [x_-, [x_+, x_-]] + [x_-, [x_-, x_+]] = 0 \]

Finally we define \( U_0 \) as the orthogonal complement of \( U_+ \oplus U_- \) in \( U \) wrt \( B_g \).

**Lemme III. 12**

1. \( U_0, U_+, U_- \) are ideals in \( U \).
2. \( [u_0, e_-] = [u_0, e_+] = 0 \)
3. \( [u_-, e_0] = [u_-, e_+] = 0 \)
4. \( [u_+, e_0] = [u_+, e_-] = 0 \).
Proof:

(1) Recall that \( \text{ad}(z) \) preserves \( \mathbb{C}^+ \) and \( \mathbb{C}^- \).

Thus if \( z \in U \), \( x, \bar{r} \in \mathbb{C}^+ \):

\[
\text{ad}(z)([x, \bar{r}]) = [\text{ad}(z)x, \bar{r}] + [x, \text{ad}(z)\bar{r}]
\]

Where \( \frac{e^z - 1}{z} \) is also in \( U \) and so is \( U^- \).

If now \( z \in U \) and \( x \in U \) is orthogonal to \( \theta \), we have \( \forall \ Y \in \mathbb{C}^+ + \mathbb{C}^- \):

\[
B(\text{ad}(z)x, Y) = -B(x, \text{ad}(z)Y)
\]

\[
= 0
\]

hence \( u_0 \) is an idempotent on \( U \).

(2) \( [u_0, \mathbb{C}^-] = 0 \): we know that \( \text{ad}(z) \)
preserves \( \mathbb{C}^- \), in particular \( [u_0, \mathbb{C}^-] \subset \mathbb{C}^- \).
Now pick $z \in u_0$, $x \in e_-$.

$$B([z, x], y) = B(z, [x_1, y]) = 0$$

Since $u_0$ is orthogonal to $u_-$. But $B$ is non-degenerate hence $[z, x] = 0$.

The same argument gives $[u_0, e_+] = 0$.

(3) Let $z \in u_-$, $x \in e_0 \pm e_+,$

$$B([z, x], y) = [u_+, e_0] = \left[ \begin{array}{c} e_+ \pm e_+ \end{array} \right]$$

$$[u_+, e_0] = \left[ \begin{array}{c} e_+, e_0 \end{array} \right]$$

$$C [e_+, [e_+, e_0]] = 0$$

Jacobi (by lemma III, 10.1.4)

and $[u_+, e_-] = \left[ \begin{array}{c} e_+, e_- \end{array} \right]$}

$$C [e_+ [e_+, e_-]] = 0$$

Jacobi (by lemma IV, 10.1.3)
Corollary III.13

(1) \( e_0 \) is an abelian ideal in \( \mathcal{Y} \).

(2) \( u_0 \oplus e_0, u_- \oplus e_- u_+ \oplus e_+ \)

are ideals in \( \mathcal{Y} \) and pairwise orthogonal.

Proof:

(1) is lemma III.10. (2)

(2) We know that \( e = e_0 \oplus e_- \oplus e_+ \)

and \( u = u_0 \oplus u_- \oplus u_+ \) are orthogonal decompositions. The orthogonality statement follows then from the fact that \( \mathcal{Y} = u + e \)

is an orthogonal decomposition.

\( u_0 \oplus e_0 \): We know (lemma III.10) that \( e_0 \) is an ideal in \( \mathcal{Y} \) and (lemma III.12) that \( u_0 \) is an ideal in \( u \).
It remains to check what happens with \([u_0, e]\). Now any \(y(u)\) preserves the decomposition \(e_0 \cdot e_+ \cdot e\) and hence \([u_0, e_0] = e\). Furthermore \(\Xi.12\ (e)\) gives \([u_0, e_+ e] = 0\), which implies \([u_0, e] = e_0\) and proves that \(u_0 + e_0\) is an ideal in \(g\).

\[u_0 + e_0\]

We compute \([u_0 + e_0, y]\):

\[\begin{align*}
[u_0 + e_0, u] &\subseteq [u_0 - u] + [e_0 - e] \\
&\subseteq u_0 - e \quad \text{(\(\Xi.12\ (11)\))}
\end{align*}\]

\([u_0 + e_0, e] \subseteq [u_0 - e] + [e_0, e]

But \([u_0, e] \subset [u_0 - e] + [u_0, e_0 + e] \quad \text{by \(\Xi.12\ (3)\)}\]
Thus: \([u_-, e] \subseteq e_-
\]

Finally: \([e_-, e] \subseteq [e_-, e_-] + [e_-, e_0] + [e_-, e_+]
\]

By definition \(u_- = [e_-, e_-]\) and

\([e_-, e_0] = 0\) by III. 10 (2) while \([e_-, e_+] = 0\)

by III. 10. (3).

The argument for \(u_+ \oplus e_+\) is the same.

Now we proceed to the definition of

the decomposition as announced in

the Theorem III. 5.

Observe that \(g_+ = u_+ \oplus e_+ \rightleftharpoons 0
\]

if \(e_+ \rightarrow 0\) in which case \(u_+ \rightarrow 0
\)

since otherwise the Killing form of \(g_+
\)

would be identically zero.
It can however happen that \( e_0 = 0 \) but \( u_0 \to 0 \) in which case \( y_0 = u_0 \) is not a good definition since the restriction of \( \Theta \) to \( y_0 \) would be the identity. Thus we proceed as follows by distinguishing cases:

(1) \( e_0 \neq 0 \): we define

\[
\begin{align*}
    y_0 &= u_0 + e_0, \\
    y_- &= u_0 + e_-, \\
    y_+ &= u_0 + e_+.
\end{align*}
\]

and the involution

\[
\Theta : = \Theta \vert_{y_0}, \quad \mu \in \{0, +, -\}.
\]

Then \( \Theta \mu \) is an involution on \( y_\mu \).

(2) \( e_0 = 0, \ e_- \neq 0 \):

\[
\begin{align*}
    y_0 &= 0, \\
    y_- &= u_0 \oplus u_- \oplus e_-, \\
    y_+ &= u_0 \oplus e_+.
\end{align*}
\]
and $g_\pm$ are just obtained by restricting $g_\pm$ to $g_\pm$.

(3) $e_0 = 0, e_- = 0$

Then $g_0 = 0, g_- = 0$ and

$g_+ = u_0 + u_+ + e_+$.

Now we have to verify that $g_0, g_-, g_+$ have the desired type.

First $g_0 = u_0 + e_0$ is of Euclidean type since by Lemma 14.10 $e_0$ is an abelian ideal in $g$ hence in $g_0$.

We know by definition that $g_-$ is negative definite on $e_-$ and positive definite on $e_+$. What remains to be seen is that the Killing form of $g_+$
coincides with the restriction of the Killing form of \( g \) ! This follows from the following general fact:

Lemma III.14:

If \( g \) is a Lie algebra and \( T \subset g \) an ideal we have: \( B_T = B_g / T \times T \).

This is left as an exercise in linear algebra.

Finally we also leave as an exercise to verify that in each case \( u_0, u_-, u_+ \) are compactly embedded in rep. \( g_0, g_-, g_+ \).
So far we have introduced three categories of objects:

1. Symmetric spaces $M \equiv \mathfrak{m}$

2. Riemannian symmetric pair $(G, \mathfrak{k})$

3. Orthogonal symmetric Lie algebra $(\mathfrak{g}, \Theta)$.

For OSL's we proved a decomposition theorem for effective OSL's. If now $(G, \mathfrak{k})$ is a riemannian symmetric pair and $\mathfrak{g} = \text{Lie} G$, $\Theta = \mathfrak{d}$ for the associated OSL, then:

$$Z(\mathfrak{g}) \cap \mathfrak{k} = \text{Lie} (Z(G) \cap \mathfrak{k})$$

and thus $(\mathfrak{g}, \Theta)$ is effective if $Z(G) \cap \mathfrak{k}$ is discrete. Accordingly
we define

\textbf{Def. III.15} \( G S P (G, K) \) is effective if \( Z(G) \cap K \) is discrete.

Now we observe the following simple

\textbf{Lemma III.16.}

Let \( G = IS(M) \), \( K = Stab_6(0) \).

If \( N \triangleleft G \) is a normal subgroup of \( G \) contained in \( K \), then \( N = e \).

In particular \( (G, K) \) is effective.

\textbf{Proof:} We have \( gKg^{-1} = Stab_6(x_0) \)

and thus if \( N \triangleleft K \) and is normal in \( G \),

we get \( N \cap gKg^{-1} = \cap Stab_6(m) = (e) \)

\( \forall g \in G \).

Observe now that \( Z(G) \) any subgroup of \( Z(G) \) is normal in \( G \) and hence

\( Z(G) \cap K = (e) \).
Def. III.17: An effective RSP \((G, K)\) is said of compact, non-compact, Euclidean type if the corresponding OS2 is.

In view of lemma III.16, the following definition then makes sense:

Def. III.18: A Riemannian symmetric space \(M\) is said of compact, non-compact, Euclidean type if the corresponding RSP \((IS(\mathbb{H}), Stab(0))\) is \(IS(\mathbb{H})\).

With the help of some basic Lie theory, we can now globally. Than III.9:
Thm III.45. Let $M$ be a simply connected Riemannian symmetric space. Then $M$ is the Riemannian product $M_0 \times M_- \times M_+$ of symmetric spaces of non-compact, compact and non-compact type.

Proof: Let $G = \text{Is}(M)^0$, $K = \text{Stab}_G(o)$,

$G = K \cdot g_o$ and $G = \text{Lie} G$, $0 = D_0$,

the associated Cox. Since $(G, B)$ is effective we can apply Thm III.4 and

get $g = g_0 \oplus g_- \oplus g_+$

the decomposition into types.

Let $G_0, G_-, G_+$ be the Lie subgroups of $G$

corresponding to $g_0, g_-, g_+$. 
We observe that $G_0, G_-, G_+$ are normal subgroups of $G$ and that all their pairwise intersections are discrete.

Now observe: $G_0 \supseteq G_-$

$$[G_0, G_2] \subseteq G_0 \cap G_2 \neq 0$$

and since $[G_0, G_2]$ is connected (Why?) and $G_0 \cap G_2$ discrete, we can write

$$[G_0, G_2] = (e) \forall \neq 0.$$ 

Thus the map

$$G_0 \times G_2 \times G_+ \rightarrow G$$

$$(x, y, z) \mapsto x \cdot y \cdot z$$

is a homomorphism which on the level of Lie algebras induces:

$$g_0 \times g_+ \times g_+ \xrightarrow{dc} g$$

$$(x, t, z) \mapsto x + t + z$$
an isomorphism. Hence by classical Lie theory we got that the lift of $\tilde{\gamma}$ to the universal covering:

$$\tilde{\gamma} : \tilde{G}_0 \times \tilde{G}_- \times \tilde{G}_+ \to \tilde{G}$$

is a Lie group isomorphism, which at the level of Lie algebras induces $\text{Det}$. Let $p : \tilde{G} \to G$ be the covering homomorphism. Then

$$\tilde{G} \xrightarrow{(p^{-1}(K))^0} \tilde{G} \xrightarrow{p^{-1}(K)} G \to M$$

is a connected covering of $M$ and since $M$ is simply connected this implies

$$\tilde{p}^{-1}(K) = (p^{-1}(K))^0$$

is connected. Thus $\tilde{p}^{-1}(K) \subset \tilde{G}$ is
the Lie subgroup associated to \( \mathfrak{g} < \mathfrak{g} \).

Let then \( K_0, K_-, K_+ \) be the Lie subgroups of \( \widetilde{G}_0, \widetilde{G}_-, \widetilde{G}_+ \) corresponding to \( \mathfrak{g}_0 < \mathfrak{g}_0, \mathfrak{g}_- < \mathfrak{g}_-, \mathfrak{g}_+ < \mathfrak{g}_+ \). Then

\[ K_0 \times K_- \times K_+ \text{ is the Lie subgroup of } \widetilde{G}_0 \times \widetilde{G}_- \times \widetilde{G}_+ \text{ corresponding to } \mathfrak{g}_0 \times \mathfrak{g}_- \times \mathfrak{g}_+ \]

and since \( \widetilde{\eta} \) is a Lie group isomorphism this implies \( \widetilde{\eta}(K_0 \times K_- \times K_+) = \widetilde{\eta}(K) \)

in particular \( K_0, K_-, K_+ \) are closed subgroups of \( \widetilde{G}_0, \widetilde{G}_-, \widetilde{G}_+ \). One version that \((\widetilde{G}_0, K_0), (\widetilde{G}_-, K_-), (\widetilde{G}_+, K_+)\)

are riemannian symmetric pairs by lifting \( \sigma \) to \( \widetilde{G} \). Finally the diffeomorphism \( \widetilde{\eta} \) induces a \( \widetilde{\eta} \)-equivariant diffeomorphism

\[ \widetilde{G}_0 \times \widetilde{G}_- \times \widetilde{G}_+ / K_0 \times K_- \times K_+ \rightarrow \widetilde{G} / \widetilde{K} = \mathcal{M}. \]