II. Mathematics

Basic Theory of Riemannian Symmetric Spaces.

Content:

II. 1. Isometries and the isometry group.

II. 2. Geodesic symmetries

(lead to loc. symm + se \implies \text{globally symm.})

II. 3. Transvections and II transport.

(transvections realize the II transport)

II. 4. Lie group viewpoint.

(Lie groupChar on b = commonion symmetric pairs (\mathfrak{g}, \mathfrak{a}))
II.5. Exponential map and geodesics.

II.6. Totally geodesic submanifolds.

(algebraic char.)

II.7. Examples.

II.8. Decomposition of symmetrized space.

- non compact type x Eucl. type x compact type.

II.9. Curvature

II.10. Semisimple Lie groups.

II.11. Duality theory.
II. 1. Isometric and the
isometry group.

Recall that a riemannian metric on a smooth manifold $M$ is a map $g$ which to every $x \in M$ associates a scalar product $g_x$ on the tangent space $T_x M$ with the following smoothness property:

for every coordinate chart

$$
\varphi : U \rightarrow \mathbb{R}^n
$$

on $M$ and $1 \leq i, j \leq n$,

the function:

$$
U \rightarrow \frac{\text{Sym}(\mathbb{R}^n)}{\mathbb{R}}
$$

$$
x \mapsto g_x \left( \overline{\left( \frac{\partial \varphi^i}{\partial x^j} \right)}(x), \overline{\left( \frac{\partial \varphi^j}{\partial x^i} \right)}(x) \right)
$$

is smooth.
The length \( l(c) \) of a smooth path \( c : [a, b] \rightarrow M \) is by definition by

\[
l(c) = \int_a^b \sqrt{g(c'(t), c'(t))} \, dt
\]

where \( \dot{c}(t) \in T_{c(t)} M \) is the tangent vector of \( c \) at \( t \).

If \( M \) is connected,

\[
d(x, y) := \inf \{ l(c) : c : [a, b] \rightarrow M \}
\]

is a smooth path with \( c(a) = x \),
\[
c(b) = y \}
\]
defines a distance, it is called the
\( \text{associated} \) Riemannian distance.

A Riemannian manifold \((M, g)\) will be a smooth manifold \( M \) endowed with a Riemannian metric \( g \).
Let \((M,g)\) and \((N,h)\) be Riemannian manifolds.

**Def. II.1** A Riemannian isometry is a diffeomorphism \(f : M \rightarrow N\) such that

\[
f^* h = g,
\]

namely,

\[
h(f(w)) \langle D_p f(u), D_p f(v) \rangle = g(u,v)
\]

\(\forall p \in M, \forall u, v \in T_p M.\)

It follows that \(f\) preserves the Riemannian distances. The converse is \(\sim\) Thm. 1.

**Thm II.2** (Hilbert Thm I.11.1)

Let \((M,g)\) be a Riemannian manifold and \(d : M \times M \rightarrow [0, \infty)\) the Riemannian distance. The following properties of a self map \(\varphi : M \rightarrow M\) are equivalent:
- \[ \text{II-6-} \]

(1) \( \varphi \) is a riemannian isometry.

(2) \( \varphi \) is a distance preserving bijection.

We will call isometry of \((M, \varphi)\) any self map \( \varphi : M \rightarrow M \) satisfying the equivalent conditions of Theorem II.2.

Contrary to isometries of general metric spaces, riemannian isometries are determined by local data:

**Lemma II.3 (Hc1) lemma I.11.12)**

Let \( \varphi_1, \varphi_2 : (M, \varphi) \rightarrow (N, \psi) \) be isometries between riemannian manifolds and assume \( M \) is connected. Assume that for some \( p \in M \) we have

(1) \( \varphi_1 \circ \psi_1 = \psi_2 \circ \psi_1 \)

(2) \( D_p \varphi_1 = D_p \varphi_2 \)

Then \( \varphi_1 = \varphi_2 \).
In case $(M,g)$ is complete, recall the following concept from Riemannian geometry:

**Normal neighborhood:**

Let $\text{Exp}_p : T_p M \to M$ be the exponential map at $p$. A normal neighborhood is an open set $U \ni p$ such that $U = \text{Exp}_p(N_0)$ where $N_0 \ni p$ is a star-shaped open set and

$$\text{Exp}^{-1}_p : N_0 \to U$$

is a diffeomorphism.

**Proof of Lemma 5.3** The map $f = f_2 f_1 : M \to M$ is a diffeomorphism satisfying $f(p) = p$, $D_p f = \text{Id}_M$. The set $S = \{ q \in M : f(q) = q, \ D_q f = \text{Id}_q \}$ is closed and $S \ni p$. We claim $S$ is open which together with $M$ connected implies $S = M$. 
Let \( g \in S \) and \( U = \text{Exp}_q (N_0) \) a normal neighborhood of \( g \). Since \( f \) is a riemannian isometry we have \( U \subset T_q M \) and \( f \) is \( L \) with \( t \in N_0 \):

\[
f (\text{Exp}_q (t u)) = \text{Exp}_q (t f_q (u)) = \text{Exp}_q (t u)
\]

which implies \( f |_U = I_d \), hence \( U \subset S \). □

Now let us discuss \( \mathfrak{Is} f M \).

Let \( \mathfrak{Is} (M) \) be the group of isometries of \( M \), the group law being given by composition of maps. Recall that the compact open topology on \( \mathfrak{Is} (M) \) is given by the subbasis of open sets:

\[
W (C, U) = \{ f \in \mathfrak{Is} (M) : f (C') \subset U \}
\]

where \( C \) is compact and \( U \) is open.
In the notes there is an outline of the following theorem; we refer the
author to the notes.

Thm 1.4.

Is(H) with compact open topology is a locally compact group acting continuously
on H.

This is based on the following facts:

(1) on Is(H) the topology of pointwise convergence and the compact open
topology coincide.

(2) Let \((f_n)_{n \geq 1}\) be a sequence in Is(H) and let \(\mathfrak{P} \in \mathfrak{P}
\). Then \((f_n)_{n \geq 1}\) admits a convergent subsequence.
In particular $\text{Stab}(p_0)$ is compact. Now using lemma II.3 we deduce that

$$\text{Stab}(p_0) \rightarrow O(T_{p_0}M)$$

$$f \mapsto D_{p_0}f$$

is an injective (cont.) homomorphism which shows $\text{Stab}(p_0)$ is isomorphic to a closed subgroup of $O(T_{p_0}M)$.

II.2. Geodesic symmetry.

We turn now to a definition of locally symmetric space that is as synthetic as possible, that is without reference to the smooth structure.

Let $(M,g)$ be a Riemannian manifold.
Def. II.5

(1) If is locally symmetric if for every \( p \in M \), there is a (normal) neighborhood \( U \) of \( p \) and an isometry

\[ S_p : U \to U \]

with

(a) \( S_p = I_u \)

(b) \( p \) is the only fixed point of \( S_p \) in \( U \).

(2) \( M \) is globally symmetric if \( S_p \) can be extended to an isometry of \( M \).

In the following lemma we get additional info on these symmetries:

Lemma II.6 Let \( p \in U \subset M \), where

\( U = \text{Exp}_p (N) \) is a normal neighborhood of \( p \);

let \( \delta \) be an isometry satisfying
Let $S_p : \mathcal{U} \to \mathcal{U}$ be on some $\mathcal{U}$ such that $p$ is the only fixed point of $S_p$.

TF+E \quad (1) \quad S_p^2 = \text{Id} \\
               (2) \quad D_p S_p = -\text{Id}.

Proof:

(2) $\Rightarrow$ (1) \quad $D_p (S_p^2) = \text{Id}$ and hence by lemma 5.3, $S_p^2 = \text{Id}$.

(1) $\Rightarrow$ (2) \quad From $S_p^2 = \text{Id}$ we conclude
               \quad $D_p S_p = -\text{Id}$. Hence $D_p S_p$ is diagonalizable with eigenvalues $1, -1$. If there is $v \in T^*_p \mathcal{U}$ with $D_p S_p (v) = v$, then $S_p (\text{Exp}_p t v) = \text{Exp}_p t v$ for $t \in \mathbb{N}$, a contradiction.
II-12.

It follows then from Lemma II.3 that if \((M, g)\) is connected locally symmetric and \(Sp : U \to U\) is a local symmetry that admits an extension to an isometry \(f_M\), this extension is unique.

Let \((M, g)\) be connected globally symmetric. We have seen that \(Is(M)\) with compact open topology is a locally compact group, thus in particular a topological group.

Let \(Is(M)^0\) be the connected component of \(Is(M)\). Then (except \(Is(M)^0\) is a closed normal subgroup of \(Is(M)\) and our next goal will be to show that it acts transitively on \(M\). To this end we will need the following lemma:
Lemma III. 7 (M, g) connected globally symmetric is ample.

Proof: By the Hopf - Riesz theorem it suffices to show that any geodesic segment

\[ \gamma : [a, b) \to M \]

can be extended to

Let \( a < b \) and \( \gamma : (a, b) \to M \) a geodesic segment. We will show that it can be extended to a geodesic segment

\[ (a, b + \frac{b-a}{2}) \]

this will readily imply that it can be extend to \( \mathbb{R} \) and implies completeness by the Hopf - Riesz theorem.

So let

\[ p = \delta \left( \frac{3b+a}{4} \right) \]

\[ a \quad \gamma \quad \frac{b}{3b+a} \]
Let $S_\rho$ be the symmetry at $\rho = \sqrt[4]{\frac{3b+\tilde{a}}{4}}$ and define the geodesic

$$\eta: \left(\frac{b+\tilde{a}}{2}, b + \left(\frac{b+\tilde{a}}{2}\right)\right) \rightarrow M$$

$$t \rightarrow S_\rho \left(\frac{3b+\tilde{a}}{2} - t\right)$$

Then

$$\eta \left(\frac{3b+\tilde{a}}{4}\right) = \eta \left(\frac{3b+\tilde{a}}{4}\right) = \rho$$

$$\eta \left(\frac{3b+\tilde{a}}{4}\right) = \partial \sigma \left(\frac{3b+\tilde{a}}{4}\right) = \eta \left(\frac{3b+\tilde{a}}{4}\right)$$

$$\eta \left(\frac{3b+\tilde{a}}{4}\right) = \eta \left(\frac{3b+\tilde{a}}{4}\right)$$

hence $$\eta \left(\frac{b+\tilde{a}}{2}, b \right) = \eta \left(\frac{b+\tilde{a}}{2}, b \right)$$

and provide the desired expansion.

We can now take the first step towards our goal:
Lemma II. 8 \((M, g)\) is connected globally
symmetric than \(\Gamma_g (M)\) acts transitively
in \(M\).

Proof: Let \(p, q \in M\), \(d(p, q)\) the
(Riemannian) distance and (by ampleness)
\[
y : \left[0, d(p, q)\right] \rightarrow M
\]
a geodesic connecting \(p\) to \(q\). Take
\(m = y \left(\frac{d(p, q)}{2}\right)\) , then
\[
\text{sm \; y(t)} = y \left(d - t \right)
\]
and hence \(\text{sm \; y(t)} = q\). \(\Box\)

The following is left as an exercise:

Lemma III. 9 \((M, g)\) connected globally
symmetric; \(p \in M\), \(K = \frac{\text{Vol} \left(\frac{1}{2} \text{sm}(p)\right)}{\text{Vol}(M)}\).

Then the orbit map

\[
\text{Is \; M} / K \rightarrow M
\]
is a homomorphism.
This will be used to show:

Lemma 10. Under the same assumptions, the map \( M \rightarrow \text{Is}(M) \)
\( g \# \rightarrow g^\# \)
is continuous.

Proof. First an observation of general interest: let \( p \in M \), \( g \in \text{Is}(M) \) then:

\[
S_{gp} = g S_p g^{-1}.
\]

Indeed,

\[
S_{gp} (g S_p g^{-1}) (gp) = gp
\]

and:

\[
D_{gp} (g S_p g^{-1}) = (D_{gp} g S_p g^{-1}) D_{gp} = (1_{gp} g S_p g^{-1}) D_{gp} = -D_{gp} S_p g S_p g^{-1} = -Id_{gp},
\]

which by lemma II. 6 implies

\[
g S_p g^{-1} = S_{gp}.
\]
Consider now the commutative diagram:

\[ 
\begin{array}{ccc}
  T & \longrightarrow & S_p = g S_p T^{-1} \\
  g P \uparrow & & \uparrow \quad Is(M) \\
  M \uparrow & & \uparrow \\
  g K \quad Is(M) \quad \downarrow \\
  & & \downarrow \\
  & & Is(M) \quad g \\
\end{array}
\]

Since \( g t \rightarrow g s_t j^{-1} \) is obviously cont. and \( Is(M) \rightarrow M \)
and descends to \( Is(M) / K \), by the definition of quasihom topology,

\[ Is(M) / K \longrightarrow Is(M) \]

\[ g K \rightarrow g s_t j^{-1} \]
is continuous. Since \( Is(M) / K \rightarrow M \)
is a homeomorphism, the map \( M \rightarrow Is(M) \)
is therefore continuous. \[ \square \]

We are now ready to prove:

Proposition II. If \( M \) is connected and globally symmetric, then there exists a transitive action on \( M \).

Proof. It follows from the preceding lemma that \( \mathbb{I} \times M \to \mathbb{I} \mathcal{S}(M) \)

\[ (b, p) \mapsto S_b \circ S_p \]

is continuous. Its image contains \( S_p = \text{id} \)

and hence \( S_b \circ S_p \in \mathbb{I} \mathcal{S}(M) \) \( \forall b, p \in M \).

Since \( M \) is connected, \( \text{Now let } p, q, q' \in M \)

\( f : [0,1] \to M \) a geodesic connecting \( p \)

\( d \).

Then:

\[ S_f \left( \frac{d}{2} \right) \circ S_f \left( \frac{d}{2} \right) = 1 \]

and \( (S_f \left( \frac{d}{2} \right) \circ S_f \left( \frac{d}{2} \right))(1) = 1. \) \[ \square \]
Here is an interesting corollary:

**Corollary 2.12** Let \((M, g)\) be connected globally symmetric, \(p \in M\) and \(K = \text{Stab}(p)\). Then \(K\) meets every connected component of \(\text{Is}(M)\), in particular \(\text{Is}(M)^0\) is open of finite index in \(\text{Is}(M)\).

**Proof:**

Let \(g \in \text{Is}(M)\); since \(\text{Is}(M)^0\) acts transitively on \(M\), there exists \(g_0 \in \text{Is}(M)^0\) with \(g p = g_0 p\), hence there is \(k \in K\) with \(g = g_0 k\). This means that \(K\) meets every closed \(\text{Is}(M)^0\)-orbit, and proves the first assertion. Thus the hom. \(\alpha : K \rightarrow \text{Is}(M)/\text{Is}(M)^0\) is surjective. But \(\text{Id} \in K^0\), hence \(K^0 \subset \text{Is}(M)^0\) and hence \(K\) factor...
\[
\text{Thm II. 12}
\]

Let \( M \) be connected, globally symmetric. Then \( \mathbb{I}_r(M) \) has a compatible smooth structure. It acts smoothly on \( M \) and the orbit map \( \mathbb{I}_r(M)/K \to M \) is a diffeomorphism.
Proof:

We already knew that $\mathbf{N}$ acquires a smooth structure via

$$\mathbf{N} \rightarrow O(T_p M)$$

$$g \mapsto \frac{\partial g}{\partial x}$$

Let $U = \text{Exp}_p(N_i)$ be a normal neighborhood of $p$. Then:

$$\phi: U \rightarrow T_p M$$

$$\text{Exp}_p u \mapsto \sum_{c=0}^{\infty} \frac{1}{c!} \text{Exp}_p \left( \frac{u}{c} \right)^c$$

is continuous.

Let $\pi: T_p M \rightarrow M$

$$g \mapsto g \cdot p$$

Then: $\pi \circ \phi(\text{Exp}_p(u)) = \sum_{c=0}^{\infty} \frac{1}{c!} \text{Exp}_p \left( \frac{u}{c} \right)^c \circ p$

$$\pi \circ \phi(\text{Exp}_p(u)) = \text{Exp}_p(u)$$

$$\text{Exp}_p \left( \frac{u}{c} \right) \circ \text{Exp}_p(u)$$
Thus: \[ \Phi: U \rightarrow \pi^{-1}(U) \rightarrow U \]

and \( \Phi = \text{id}_U \), which implies that \( \Phi \) is a homeomorphism onto its image, since it has a continuous inverse.

Hence the map

\[ \gamma: U \times K \rightarrow \pi^{-1}(U) \]

\( (u, k) \rightarrow \Phi(u) \cdot k \)

is bijective and continuous and it has a continuous inverse given by:

\[ \pi^{-1}(U) \rightarrow U \times K \]

\[ g \rightarrow (g \cdot p, \phi(g \cdot p \cdot g)) \]

Thus: \( \Phi': \pi^{-1}(U) \rightarrow U \times K \) is a homeo between the open neighborhood \( \pi^{-1}(U) \) of \( \text{Id} \) and \( U \times K \). One obtains an atlas on \( \pi^{-1}(U) \) by covering it by left translates of \( \pi^{-1}(U) \). One has then to check compatibility.