II. ~~Generalities~~ Basic Theory of Riemannian Symmetric Spaces.

Content:

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II. 2. Geodesic symmetries
(leads to loc. symm + s.c ⟹ globaly symm.)

II. 3. Transvections and // transport.
(transvections realize the // transport).

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(Lie grp Charact. by riemannian symmetric pairs (G, σ))

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II. 5. Exponential map and geodesics.

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non compact type × Eucld. type × compact type.

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II. 1. Isometries and the isometry group.

Recall that a riemannian metric on a smooth manifold M is a map g which to every $x \in M$ associates a scalar product g_x on the tangent space $T_x M$ with the following smoothness property:

for every coordinate chart

$$\varphi : \underset{\underset{M}{\cap}}{U} \longrightarrow \mathbb{R}^n$$

and $1 \leq i, j \leq n$,

the ~~function~~:

$$\begin{aligned} U &\longrightarrow \cancel{Sym_n(\mathbb{R})}\ \mathbb{R} \\ x &\longmapsto g_x\left((D_x \varphi)^{-1}(e_i), (D_x \varphi)^{-1}(e_j) \right) \end{aligned}$$

is smooth.

The length $\ell(c)$ of a smooth path

$$c : [a, b] \longrightarrow M$$

is by definition

$$\ell(c) = \int_a^b \sqrt{g_{c(t)}(\dot{c}(t), \dot{c}(t))}\, dt$$

where $\dot{c}(t) \in T_{c(t)}M$ is the tangent vector of c at t.

If M is connected,

$$d(x,y) := \inf \{ \ell(c) : c : [a,b] \to M \text{ is a smooth path with } c(a) = x, c(b) = y \}$$

defines a distance: it is called the (associated) Riemannian distance.

A riemannian manifold (M, g) will be a smooth manifold M endowed with a riemannian metric g.

Let (M, g) and (N, h) be riemannian manifolds.

<u>Def. II.1</u> A riemannian isometry is a diffeomorphism $f : M \to N$ such that

$$f^* h = g$$

namely:
$$h_{f(p)}(D_p f(u), D_p f(v)) = g_p(u, v)$$

$\forall p \in M$, $\forall u, v \in T_p M$.

It follows that f preserves the riemannian distances. The converse is a thm.:

<u>Thm II.2 (Hel Thm I.11.1)</u>

Let (M, g) be a riemannian manifold and $d : M \times M \to [0, \infty)$ the riemannian distance. The following properties of a self map $\varphi : M \to M$ are equivalent:

(1) φ is a riemannian isometry.

(2) φ is a distance preserving bijection.

We will call isometry of (M, g) any self map $\varphi : M \to M$ satisfying the equivalent conditions of Thm II.2.

Contrary to isometries of general metric spaces, riemannian isometries are determined by local data:

<u>Lemma II.3</u> ([Hel] lemma I.11.12)

Let $f_1, f_2 : (M, g) \to (N, h)$ be isometries between riemannian manifolds and assume M is connected. Assume that for some $p \in M$ we have

(1) $f_1(p) = f_2(p)$

(2) $D_p f_1 = D_p f_2$ Then $f_1 = f_2$.

In case (M, g) is complete this is immediate.

Recall the following concept from riemannian geometry:

Normal neighborhood:

let $Exp_p : T_pM \to M$ be the exponential map at p. A normal neighborhood is an open set $U \ni p$ such that $U = Exp_p(N_0)$ where $N_0 \ni 0$ is a star shaped open set and

$$Exp_p|_{N_0} : N_0 \to U$$

is a diffeomorphism.

Proof of lemma II.3 The map $f := f_2^{-1} f_1 : M \to M$ ~~satisfies~~ is an isometry satisfying $f(p) = p$, $D_p f = Id$. The set

$$S = \{ q \in M : f(q) = q \, , D_q f = Id \} \text{ is}$$

closed and $S \ni p$. We claim S is open which together with M connected implies $S = M$.

Let $q \in S$ and $\mathcal{U} = Exp_q(N_0)$ a normal neighborhood of q. Since f is a riemannian isometry we have $\forall v \in T_qM$ and ~~tt~~ $t \in \mathbb{R}$ with $tv \in N_0$:

$$f(Exp_q(tv)) = Exp_q(t D_q f(v))$$
$$= Exp_q(tv)$$

which implies $f|_{\mathcal{U}} = Id_{\mathcal{U}}$, hence $\mathcal{U} \subset S$. $\boxed{}$

~~Now let us discuss Is(M).~~

Let $Is(M)$ be the group of isometries of M, the group law being given by composition of maps. Recall that the compact open topology on $Is(M)$ is given by the subbasis of open sets:

$$W(C, \mathcal{U}) = \{ f \in Is(M) : f(C) \subset \mathcal{U} \}$$

where C is compact and $\mathcal{U}$ is open.

In the notes there is an outline of the following theorem; we refer the auditor to the notes.

Thm II.4.

Is(M) with compact open topology is a locally compact group acting continuously on M.

This is based on the following facts

(1) On Is(M) the topology of pointwise convergence and the compact open topology coincide.

(2) Let $(f_n)_{n\geq 1}$ be a sequence in Is(M) and p_0 s.t. $(f_n(p_0))_{n\geq 1}$ converges.

Then $(f_n)_{n\geq 1}$ admits a convergent subsequ.

In particular $Stab(p_0)$ is compact.
Now using lemma II.3 we deduce that

$$\begin{aligned} Stab(p_0) &\longrightarrow O(T_{p_0}M) \\ f &\longmapsto D_{p_0} f \end{aligned}$$

is an injective (cont.) homomorphism which shows $Stab(p_0)$ is isomorphism to a closed subgroup of $O(T_{p_0}M)$.

II.2. Geodesic symmetries.

We turn now the a definition of locally symmetric space that is as synthetic as possible, that is without reference to the smooth structure.

Let (M, g) be a Riemannian manifold.

Def. II.5

(1) M is locally symmetric if for every $p \in M$ there is a (normal) neighborhood U of p and an isometry

$$S_p : U \longrightarrow U$$

with

(a) $S_p^2 = Id$

(b) p is the only fixed point of S_p in U.

(2) M is globally symmetric if S_p can be extended to an isometry of M.

In the following lemma we get additional infos on these symmetries:

Lemma II.6 Let $p \in U \subset M$, where $U = Exp_p(N_0)$ is a normal neighborhood of p;

~~let $S_p : U \to U$ be an isometry satisfying~~

Let $S_p : \mathcal{U} \to \mathcal{U}$ be an isometry such that p is the only fixed point of S_p.

TFAE (1) $S_p^2 = Id$

(2) $D_p S_p = -Id$.

Proof:

(2) $\Rightarrow$ (1) $D_p(S_p^2) = Id$ and hence by lemma II.3, $S_p^2 = Id$.

(1) $\Rightarrow$ (2): From $S_p^2 = Id$ we conclude $(D_p S_p)^2 = Id$. Hence $D_p S_p$ is diagonalizable with eigenvalues $1, -1$. If there is $v \in T_p \mathcal{N}_0$ with $D_p S_p(v) = v$, then $S_p(\mathrm{Exp}_p\, tv) = \mathrm{Exp}_p\, tv$ $\forall\, tv \in \mathcal{N}_0$ ~~and hence~~

↯ contradiction. $\boxed{↯}$

It follows then from lemma II.3 that if (M, g) is connected locally symmetric and $s_p : U \to U$ is a local symmetry that admits an extension to an isometry of M, this extension is unique.

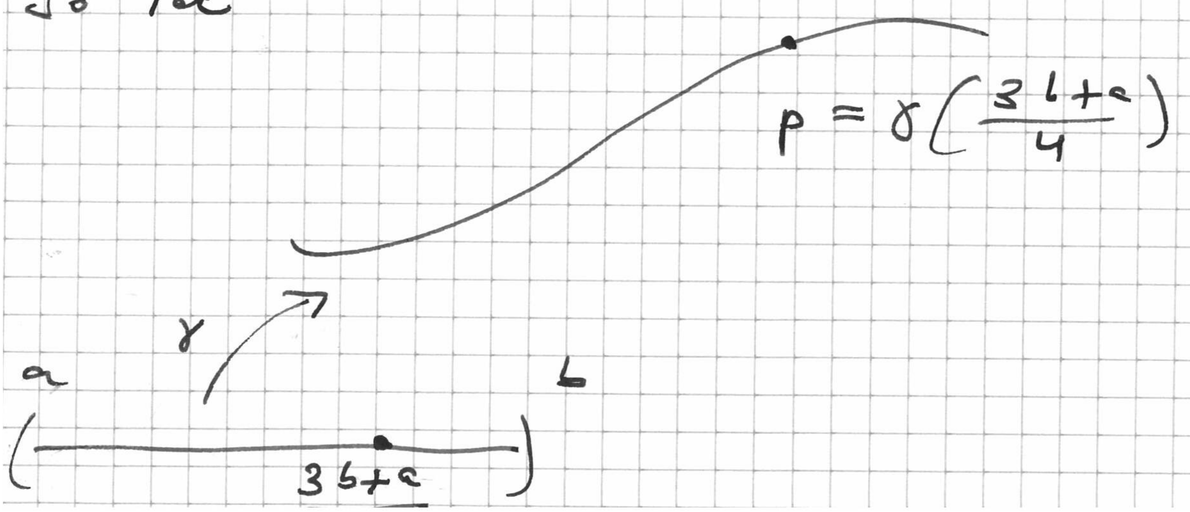
Let (M, g) be connected globally symmetric. We have seen that $Is(M)$ with compact open topology is a locally compact group, thus in particular a topological group. Let $Is(M)^0$ be the connected component ~~of~~ of $\{Id\}$. Then (exercise) $Is(M)^0$ is a closed normal subgroup of $Is(M)$ and our next goal will be to show that it acts transitively on M. To this end we will need the following lemmas:

<u>Lemma II.7</u> (M,g) connected globally symmetric is complete.

<u>Proof</u>: ~~By the Hopf-Rinow theorem it suffices to show that any geodesic segment $\gamma : [a,b) \longrightarrow M$ can be extended to b.~~

Let $a < b$ and $\gamma : (a,b) \longrightarrow M$ a geodesic segment. We will show that γ can be extended to a geodesic segment $\left(a, b + \left(\frac{b-a}{2}\right)\right)$; this will readily imply that γ can be extend to $\mathbb{R}$ and implies completeness by the Hopf-Rinow theorem.

So let



Let S_p be the symmetry at $p = \gamma\left(\frac{3b+a}{4}\right)$ and define the geodesic

$$\eta : \left(\frac{b+a}{2}, \, b + \frac{(b-a)}{2}\right) \longrightarrow M$$

$$t \longmapsto S_p\left(\gamma\left(\frac{3b+a}{2} - t\right)\right)$$

Then $\eta\left(\frac{3b+a}{4}\right) = \gamma\left(\frac{3b+a}{4}\right) = p$

$$\dot\eta\left(\frac{3b+a}{4}\right) = D_p S_p\left(-\dot\gamma\left(\frac{3b+a}{4}\right)\right)$$
$$= \dot\gamma\left(\frac{3b+a}{4}\right)$$

hence $\eta\big|_{\left(\frac{b+a}{2}, b\right)} = \gamma\big|_{\left(\frac{b+a}{2}, b\right)}$

and provides the desired extension. $\boxed{\xi}$

We can now take the first step towards our goal:

<u>Lemma II.8</u> If (M, g) is connected globally symmetric then $Is(M)$ acts transitively in M.

<u>Proof:</u> Let $p, q \in M$, $d(p,q)$ the (riemannian) distance and (by completeness)

$$\gamma : [0, d(p,q)] \longrightarrow M$$

a geodesic connecting p to q. Take

$$m = \gamma\left(\frac{d(p,q)}{2}\right)$$; then

$$s_m(\gamma(t)) = \gamma(d - t)$$

and hence $s_m(p) = q$. $\boxed{\xi}$

The following is left as exercise:

<u>Lemma II.9</u> (M, g) connected globally symmetric; $p \in M$, $K = \mathrm{Stab}_{Is(M)}(p)$.

Then the orbit map

$$Is(M)/_K \longrightarrow M \quad \text{is a homeo-}$$

This will be used to show:

<u>Lemma II.10</u>. Under the same assumptions the map

$$M \longrightarrow Is(M)$$
$$q \longmapsto s_q$$

is continuous.

<u>Proof</u>: First an observation of general interest: let $p \in M$, $g \in Is(M)$ then:

$$s_{gp} = g\, s_p\, g^{-1}.$$

Indeed, ~~$s_{gp}(gp) = gp$~~

$$(g s_p g^{-1})(gp) = gp$$

and: $$D_{gp}(g s_p g^{-1}) = (D_p g)\underbrace{(D_p s_p)}_{-Id}(D_{gp} g^{-1})$$

$$= -D_p g\, D_{gp} g^{-1} = -Id_{gp}$$

which by lemma II.6 implies

$$g s_p g^{-1} = s_{gp}.$$

Consider now the commutative diagram:

$$\begin{tikzcd}
 & gp \arrow[r, mapsto] & S_{gp} = g S_p g^{-1} \\
gp & M \arrow[r] & Is(M) \\
gK & Is(M)/K \arrow[u] \arrow[ur] & \\
 & Is(M) \arrow[u] \arrow[uur, bend right] & \\
 & g \arrow[u, phantom, sloped, "\in"] \arrow[uul, bend left] &
\end{tikzcd}$$

$$gK \;\; Is(M)/K$$

Since $g \longmapsto g S_p g^{-1}$ is obviously cont.

~~and~~ $\cancel{Is(M)/K \longrightarrow M}$

and descends to $Is(M)/K$, by the definition of quotient topology,

$$Is(M)/K \longrightarrow Is(M)$$

$$gK \longmapsto g S_p g^{-1}$$

is continuous; since $Is(M)/K \longrightarrow M$ is a homeo., the map $M \longrightarrow Is(M)$

$$gp \longmapsto S_{gp}$$

is therefore continuous. $\boxed{\Sigma}$

We are now ready to prove:

<u>Proposition II.11</u> M connected, globally symmetric, then $Is(M)^0$ acts transitively on M.

<u>Proof:</u> It follows from the preceding lemma that

$$M \times M \longrightarrow Is(M)$$
$$(p, p') \longmapsto s_p \circ s_{p'}$$

is continuous. Its image contains

$$s_p^2 = Id$$

and hence $s_p \circ s_{p'} \in Is(M)^0 \quad \forall p \in M$ since M is connected. Now let $p, q \in M$

$\gamma : [0, d] \longrightarrow M$ a geodesic connecting p to q. Then: $s_{\gamma(\frac{d}{2})} \circ s_p \in Is(M)^0$

and $\left(s_{\gamma(\frac{d}{2})} \circ s_p \right)(p) = q$. $\boxed{\Sigma}$

Here is an interesting Corollary:

<u>Corollary II.12</u> Let (M,g) be connected globally symmetric, $p \in M$ and $K = \mathrm{Stab}_{Is(M)}(p)$.

Then K meets every connected component of $Is(M)$, in particular $Is(M)^\circ$ is open of finite index in $Is(M)$.

<u>Proof</u>:

Let $g \in Is(M)$; since $Is(M)^\circ$ acts transitively on M, there exists $g_0 \in Is(M)^\circ$ with $gp = g_0 p$, hence there is $k \in K$ with $g = g_0 \cdot k$. This means that K meets every coset $Is(M)^\circ \cdot g$, $g \in Is(M)$ and proves the first assertion. Thus the hom.

$$\alpha : K \longrightarrow Is(M)/Is(M)^\circ$$
$$k \longmapsto k \cdot Is(M)^\circ$$

is surjective. But $\mathrm{Id} \in K^\circ$, hence $K^\circ \subset Is(M)^\circ$ and hence α factors to $K/K^\circ \longrightarrow Is(M)/Is(M)^\circ$. Since K is a compact Lie group K° is open of finite index in K which concludes the second part. $\boxed{}$

We now sketch a proof of the following; details can be found in Helgason, Chap IV lemma 3.2.

Thm II.12

Let M be connected, globally symmetric. Then $Is(M)$ has a compatible smooth structure. It acts smoothly on M and the orbit map $Is(M)/K \longrightarrow M$ is a diffeomorphism.

Proof:

We already know that K acquires a smooth structure via

$$\begin{aligned} K &\longrightarrow O(T_pM) \\ g &\longmapsto d_p g. \end{aligned}$$

Let $\mathcal{U} = Exp_p(N_0)$ be a normal neighborhood of p. Then:

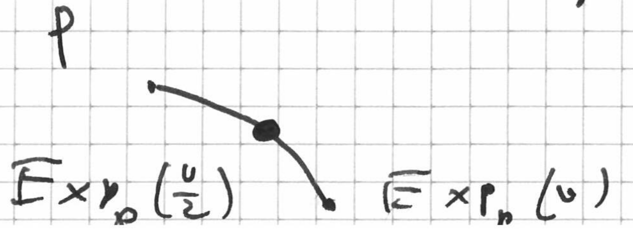
$$\begin{aligned} \phi : \mathcal{U} &\longrightarrow Is(M) \\ Exp_p u &\longmapsto s_{Exp_p(\frac{u}{2})} \circ s_p \end{aligned}$$

is continuous.

Let
$$\begin{aligned} \pi : Is(M) &\longrightarrow M \\ g &\longmapsto g\cdot p \end{aligned}$$

Then: $\pi\,\phi(Exp_p u) = s_{Exp_p(\frac{u}{2})}\, s_p\,(p)$

$= Exp_p(u)$



Thus: $U \xrightarrow{\phi} \pi^{-1}(U) \xrightarrow{\pi} U$

and $\pi \circ \phi = id_U$ which implies that ϕ is a homeomorphism onto its image, since it has a continuous inverse.

Hence the map

$$\begin{aligned} \psi : U \times K &\longrightarrow \pi^{-1}(U) \\ (u, k) &\longmapsto \phi(u) \cdot k \end{aligned}$$

is bijective continuous and it has a continuous inverse given by:

$$\begin{aligned} \pi^{-1}(U) &\longrightarrow U \times K \\ g &\longmapsto (g \cdot p, \phi(gp)^{-1} \cdot g). \end{aligned}$$

Thus: $\psi^{-1} : \pi^{-1}(U) \longrightarrow U \times K$ is a homeo between the open neighborhood $\pi^{-1}(U)$ of Id and $U \times K$. One obtains an atlas on $Is(M)$ by covering it by left translates of $\pi^{-1}(U)$. One has then to check compatibility.

$\boxed{\xi}$