We end this section by mentioning an important theorem relating locally symmetric and globally symmetric spaces. Its proof uses the characterization of locally symmetric spaces in terms of the riemannian curvature tensor; a good proof ~~in~~ can be found in Maubon section 3.

<u>Thm II. 13</u> Let (M, g) be ~~some~~ a connected, simply connected locally symmetric space. Then (M,g) is globally symmetric.

The converse does not hold:

<u>Example II. 14</u> Let S^n be endowed with its standard riemannian metric. We have seen that S^n is globally symmetric, each symmetry s_p, $p \in S^n$ being the restriction to S^n of an element

in $O(n+1)$. Let $\mathbb{P}^n(\mathbb{R}) := \{\pm Id\} \backslash S^n$, with the riemannian metric induced by the covering map $\pi : S^n \longrightarrow \mathbb{P}^n(\mathbb{R})$. Since s_p commutes with $\pm Id$, we get $\forall\, q \in \mathbb{P}^n(\mathbb{R})$ a global geodesic symmetry at q. Hence $\mathbb{P}^n(\mathbb{R})$ is globally symmetric but $\pi_1(\mathbb{P}^n(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$.

An important consequence of the above theorem is the following structure theorem for locally symmetric spaces: let (M, g) be locally symmetric connected complete, and $\tilde{g}$ the corresponding riemannian metric on the universal covering $\tilde{M}$. Then $M = \Gamma \backslash \tilde{M}$ where $\Gamma < Is(\tilde{M})$ is a subgroup acting prop. disc. without fixed points on $\tilde{M}$. Now $(\tilde{M}, \tilde{g})$ is connected, simply connected complete hence globally symm.

and thus diffeom. to $Is(\tilde{M})/_K$ where $K = \mathrm{Stab}_{Is(\tilde{M})}(p)$. Thus:

$$M = \Gamma \backslash {}^{Is(\tilde{M})}/_K .$$

II.3. Transvections and parallell transport.

We begin this section with a reminder from differential geometry concerning connections and parallell transport. A good reference is Do Carmo Chapter 2.

Let M be a smooth manifold; TM its tangent bundle, $\pi : TM \to M$ the canonical projection. Recall that a smooth vector field on M is a smooth map $X : M \to TM$ such that $\pi \circ X = \mathrm{id}$.

Let $\mathrm{Vect}(M)$ be the space of smooth vector fields on M. It is not only a $\mathbb{R}$-vector space, but also a $C^\infty(M)$-module via pointwise multiplication:

$$(f \cdot X)(p) := f(p)\, X(p), \quad f \in C^\infty(M), \; X \in \mathrm{Vect}(M)$$

For $f \in C^\infty(M)$, there is a well defined notion of derivative $d_p f : T_p M \longrightarrow \mathbb{R}$ which is a linear form on $T_p M$ $\forall p \in M$.

In this way $X \in \mathrm{Vect}(M)$ acts on $C^\infty(M)$ by:

$$(X f)(p) = (d_p f)(X(p)).$$

While the directional derivative of a function wrt a vector field is well defined, there is no canonical way to define the directional derivative of a vector field wrt another vector field.

Thus the concept of connection:

<u>Def. II. 15</u> A connection on M is a map $\nabla : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \longrightarrow \mathrm{Vect}(M)$

$$(X, Y) \longmapsto \nabla_X Y$$

such that

(1) ∇ is $C^\infty(M)$ - linear in X

(2) ∇ is $\mathbb{R}$ - linear in Y

(3) ∇ satisfies the Leibniz rule:

$$\nabla_X (f \cdot Y) = f \cdot \nabla_X Y + (Xf) \cdot Y.$$

The first condition implies that $(\nabla_X Y)(p)$ only depends on $X(p)$ while the third implies that if Y_1, Y_2 coincide in a neighborhood of p:

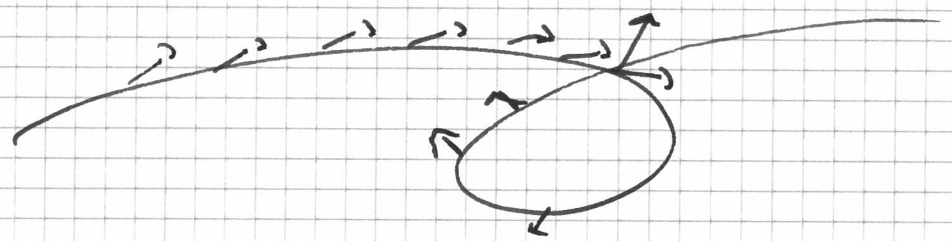
$$\nabla_X Y_1 (p) = \nabla_X Y_2 (p).$$

<u>Terminology</u>: let $I \subset \mathbb{R}$ be any interval. A map $f : I \to N$ into a smooth manifold

is smooth if it is the restriction to I of a smooth map defined on an open interval $J \supset I$.

Let $c : I \to M$ be a smooth curve.

<u>Def. II.16</u> A vector field along c is a smooth map $V : I \longrightarrow TM$ such that $V(t) \in T_{c(t)} M$.



We denote by $\mathrm{Vect}(c^* TM)$ the vector space of vector fields along c. Then we have

<u>Lemma II.17</u>. Given a smooth manifold M a connection ∇ on M and a smooth curve $c : I \to M$ there is a unique linear map

$$\frac{D}{dt} : \mathrm{Vect}(c^* TM) \longrightarrow \mathrm{Vect}(c^* TM)$$

such that

(1) $\dfrac{D}{dt}(f\cdot V) = f' V + f \dfrac{DV}{dt}$ $\quad \forall f \in C^\infty(I)$

$\forall V \in \mathrm{Vect}_c(TM)$

(2) if $V(t) = X(c(t))$, $X \in \mathrm{Vect}(TM)$

then $\dfrac{DV}{dt}(t) = \left(\nabla_{\dot c(t)} X\right)(c(t))$.

<u>Def II. 18</u> A vector field V along c is called parallel if $\dfrac{DV}{dt}(t) = 0$ $\quad \forall t \in I$.

<u>Prop. II. 19</u> Let M be a smooth manifold endowed with a connection ∇ and $c : [0,1] \to M$ a smooth curve. Then $\forall v_0 \in T_{c(0)}M$ there exists a unique vector field V along c s.t.

(1) $V(0) = v_0$

(2) $\dfrac{DV}{dt}(t) = 0$ $\quad \forall t \in [0,1]$.

Thus given $c : I \to M$ smooth and $t_0, t_1 \in I$ we get a well defined linear map $T_{c;\, t_0, t_1} : T_{c(t_0)} M \longrightarrow T_{c(t_1)} M$

which $\forall\, v \in T_{c(t_0)} M$ associates the value $V(t_1)$ of the unique v.f. along $c|_{[t_0, t_1]}$ (or $c_{[t_1, t_0]}$) and parallel along it.

Because of uniqueness of // vector fields we have $\forall\, t_0, t_1, t_2$ in I:

$$P_{c;\, t_1, t_2} \circ P_{c;\, t_0, t_1} = P_{c;\, t_0, t_2}.$$

Given now a riemannian metric $\langle\ \rangle_x \in T_x M$, we obtain a bilinear symmetric map

$$\mathrm{Vect}(M) \times \mathrm{Vect}(M) \longrightarrow C^\infty(M)$$

$$(X, Y) \longmapsto \langle X, Y \rangle$$

where $\langle X, Y \rangle (x) := \langle X(x), Y(x) \rangle_x$.

A fundamental theorem asserts then that there exists a unique connection ∇ on M satisfying:

$$(1) \quad \nabla_X Y - \nabla_Y X = [X, Y]$$

$$(2) \quad X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle .$$

called Levi–Civita connection.

~~A geometric consequence is that if~~

$$\cancel{c : I \longrightarrow (M, g)}$$

~~is a smooth curve and V_1, V_2 are parallel vector fields along c~~

Let $c : I \longrightarrow M$ be a smooth curve and $\frac{D}{dt} : Vect(c^* TM) \longrightarrow Vect(c^* TM)$.

Then (2) translates into:

$$\frac{d}{dt}\langle V_1(t), V_2(t)\rangle_{c(t)} = \left\langle \frac{DV_1}{dt}, V_2 \right\rangle_{c(t)} + \left\langle V_1, \frac{DV_2}{dt}\right\rangle_{c(t)}$$

In particular if V_1, V_2 are parallell vector fields, $t \longmapsto \langle V_1(t), V_2(t)\rangle_{c(t)}$ is constant and hence the parallell transport

$$P_{c;\, t_0, t_1} : T_{c(t_0)}M \longrightarrow T_{c(t_1)}M$$

preserves inner product.

Let now $f: M \to M$ be a diffeomorphism then it induces a linear map, the pushforward of vector fields:

$$f_* : \mathrm{Vect}(M) \longrightarrow \mathrm{Vect}(M)$$

defined by $(f_* X)(p) = (D_{f^{-1}(p)} f)(X(f^{-1}(p)))$.

$$X(f^{-1}(p)) \longleftarrow \underset{f^{-1}(p)}{\circ} \xrightarrow{\ f\ } \underset{p}{\cdot} \;:\; (D_{f^{-1}(p)} f)(X(f^{-1}(p)))$$

Recall that one can consider $Vect(M)$ as the space of Derivations of the algebra $C^\infty(M)$. Then $f_* X$ seen as a derivation is: $(f_* X)(\varphi) = X(\varphi \circ f^{-1})$

$\varphi \in C^\infty(N)$.

This implies that $X \longrightarrow f_* X$ is a Lie algebra morphism and hence

$$f_*([X,Y]) = [f_* X, f_* Y].$$

Now ~~Thus~~ if ∇ is an arbitrary connection, so is: $D_X Y := f_*^{-1}\left(\nabla_{f_* X}(f_* Y)\right)$.

and if $\nabla_X Y - \nabla_Y X = [X,Y]$

then the same holds for $D_X Y$.

The following follows then from the uniqueness of the Levi-Civita connection:

<u>Lemma II.20.</u> Let ∇ be the Levi-Civita connection of (M,g) and $f \in Is(M)$.

Then $$f_*^{-1}\left(\nabla_{f_* X}(f_* Y)\right) = \nabla_X Y \quad \forall X, Y.$$

Which can also be written like

$$\nabla_{f_* X}(f_* Y) = f_*(\nabla_X Y).$$

We are going to need an analogous formula for the derivative of vector fields along a geodesic. The situation is the following:

let $\gamma : \mathbb{R} \longrightarrow M$ be a geodesic

and $f : M \longrightarrow M$ an isometry that preserves γ, that is, there is an isometry

$$T : \mathbb{R} \longrightarrow \mathbb{R}$$

such that the diagramme:

$$\begin{tikzcd} M \arrow[r, "f"] & M \\ \mathbb{R} \arrow[u, "\gamma"] \arrow[r, "T"'] & \mathbb{R} \arrow[u, "\gamma"'] \end{tikzcd}$$

commutes. Notice that any isometry of $\mathbb{R}$ is of the form $T(t) = at+b$ where $a \in \{-1, 1\}$.

For $V \in Vect(c^*TM)$ define then

$$(f_* V)(t) = \left(D_{\gamma(T^{-1}(t))} f \right) \left(V(T^{-1}(t)) \right)$$

in analogy with the above.

Then:

<u>Lemma II.21</u> We have

$$\frac{D f_* V}{dt} = \pm f_* \left(\frac{DV}{dt} \right)$$

where we have $T(t) = \pm t + b$.

In particular, if V is $\parallel$ along γ, $f_* V$ is $\parallel$ along γ.

Let now M be globally symmetric and $\gamma : \mathbb{R} \to M$ a geodesic. Define

$$\tau_t := s_{\gamma(\frac{t}{2})} \cdot s_{\gamma(0)} \in Is(M)^0 .$$

Such isometries are called <u>transvections</u>.

We have the following important:

<u>Prop. II.22.</u>

(1) $\tau_t(\gamma(s)) = \gamma(s+t) \quad \forall s, t$

(2) $D_{\gamma(s)} \tau_t : T_{\gamma(s)} M \longrightarrow T_{\gamma(s+t)} M$

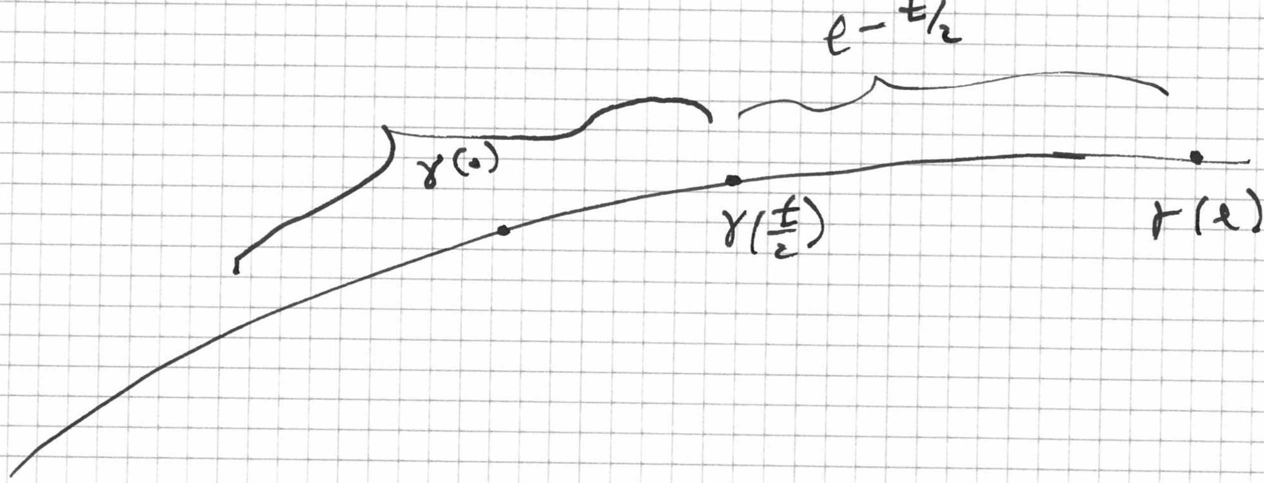
implements the $\parallel$ transport.

(3) $t \longmapsto \tau_t$ is a 1-parameter group in $Is(M)^0$.

(4) τ_t is independent of o.p. reparam. of γ.

Proof:

(1) We have $s_{\gamma(\frac{t}{2})}(\gamma(l)) = \gamma(t-l)$



Start and $\gamma(l)$ and move $2(l - t/2)$ back in time to end up at time: $l - 2(l - t/2) = t - l$.

Thus $$s_{\gamma(\frac{t}{2})}\, s_{\gamma(0)}(\gamma(s)) = s_{\gamma(\frac{t}{2})}\, \gamma(-s)$$

$$= \gamma(t+s).$$

Thus $$\tau_t(\gamma(s)) = \gamma(s+t).$$

(2) Since $s_{\gamma(l)}(\gamma(t)) = \gamma(-t+2l)$

it follows from lemma I.21 that whenever V is parallel along γ, so is $(s_{\gamma(l)})_* V$.

Now let us determine $(s_{\gamma(\ell)})_* V$ under the assumption that V is parallel. Evaluating at ℓ gives

$$\left((s_{\gamma(\ell)})_* \dot V\right)(\ell) = -V(\ell)$$

since $D_{\gamma(\ell)} s_{\gamma(\ell)} = -Id$. Thus by uniqueness of parallel v.f. we conclude:

$$(s_{\gamma(\ell)})_* V = -V$$

and hence $(\tau_t)_* V = V \quad \forall t$.

~~A computation~~

Pluging in the definition of $(\tau_t)_*$ we get:

$$(D_{\gamma(s)} \tau_t)(V(s)) = V(s+t),$$

whenever V is a parallel vector field.

But this implies that

$$D_{\gamma(s)} \tau_t = P_{\gamma;\, s,\, s+t}.$$

(3) We have

$$D_{\gamma(s)} \tau_{t_1+t_2} = P_{\gamma; s, s+t_1+t_2}$$

$$= P_{\gamma; s+t_1, s+t_1+t_2} \cdot D_{\gamma, s, s+t_1}$$

$$= D_{\gamma(s+t_1)} \tau_{t_2} \cdot D_{\gamma(s)} \tau_{t_1}$$

$$= D_{\gamma(s)} \left(\tau_{t_2} \circ \tau_{t_1} \right)$$

and hence $\tau_{t_1+t_2} = \tau_{t_2} \cdot \tau_{t_1}$.

(4). $\tau_t = s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}$. If we place the origin of γ at a instead and compute:

$$s_{\gamma(\frac{t}{2}+a)} s_{\gamma(a)} = s_{\gamma(\frac{t}{2}+a)} \circ s_{\gamma(0)} \circ s_{\gamma(0)} s_{\gamma(a)}$$

$$= \tau_{t+2a} \left(s_{\gamma(a)} s_{\gamma(0)} \right)^{-1}$$

$$= \tau_{t+2a} \left(\tau_{2a} \right)^{-1} = \tau_t . \quad \square$$

Def. II. 23. The map $\mathbb{R} \longrightarrow Is(M)^\circ$

$t \longmapsto \tau_t$

is called the one parameter group of transvections associated to the geodesic γ.

II. 4. Lie group viewpoint.

Thus we have seen how a geodesic γ in M leads to a 1-parameter group in $Is(M)^\circ$ which ~~itself~~ thus is given by the Lie group exponential of a line in the Lie algebra of $Is(M)^\circ$. In order to understand better this relationship we need to develop further the Lie group side. Recall that we have seen that M is a homog. space under $Is(M)^\circ$ with compact stability group K. We will now characterize the pairs (G, K)

consisting of a Lie group and a compact subgroup leading to symmetric space.

The central notion is

<u>Def. II.24</u> Let G be a Lie group. An automorphism $\sigma : G \longrightarrow G$ is an <u>involution</u> if $\sigma \neq Id_G$ and $\sigma^2 = Id_G$.

Here is the first step linking RSS to Riem. sym. pairs (G, σ).

<u>Prop. II.25.</u> $M \ni o$ RSS, $G = Is(M)^\circ$, $K = Stab_G(o)$

Then:

$$\sigma : G \longrightarrow G$$
$$g \longmapsto s_o g s_o$$

is an involutive autom. of G and

$$(G^\sigma)^\circ \subset K \subset G^\sigma$$

where $G^\sigma = \{ g \in G : \sigma(g) = g \}$.