Proof:

Observe that since \( \sigma _0 = \sigma _1^{-1} \),
\[
\sigma (g) = \sigma _0 \cdot g \cdot \sigma _0 = g \\
\sigma (\sigma (g)) = (\sigma (g)) \cdot \sigma (g) = \sigma (g) \cdot g = \sigma (g) \cdot g \cdot \sigma (g) = g \\
\]
Now \( \sigma (g) = g \) is equivalent to
means \( \sigma _0 \cdot g \cdot \sigma _0 = g \) which is equivalent to
\[
\sigma _0 = g \cdot \sigma _0^{-1} = \sigma _0 \\
\]
Thus, if \( g_0 = 0 \), we conclude \( g \cdot \sigma _0 \cdot g^{-1} = 0 \),
hence \( \sigma _0 \cdot g \cdot \sigma _0 = g \) that is \( g \in \mathfrak{g}^0 \), which
shows \( \mathfrak{k} \subseteq \mathfrak{g}^0 \). Finally let's show \( (\mathfrak{g}^0)^0 \) is.

Let \( V \ni 0 \) be an open neighborhood of \( 0 \) such that \( 0 \) is the only fixed point of \( \sigma_0 \) in \( V \).
Then \( U : = \{ g \in \mathfrak{g}^0 : g \cdot e \in V \} \) is an open neighborhood of \( e \) in \( \mathfrak{g}^0 \). If now
\( \mathfrak{k} \cup U \) we have \( \sigma_0 = \sigma_0 \) and hence \( g \in \mathfrak{k} \cup U \).
Thus \( U \subset K \) and therefore the open subgroup of \( G^0 \) generated by \( U \) is contained in \( K \) implying \((G^0) \cap K\).

Example III.26. Work out the following example:

1. \( M = U^2 \), \( G = T \times \{e\}^0 = T_0(3) \)

Take \( 0 = e_3 \in \mathbb{R}^2 \); then \( \mathbf{g}_0 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \).

\[ G^0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in O(2) \right\} \]

Let \( A \cdot d = 1 \).

\( K = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in SO(2) \right\} \) is

connected.
Let $G$ be a connected Lie group; $\mathfrak{g}$ its Lie algebra. For every $g \in G$, let

$$\text{Int}(g) : G \to G$$

$$u \mapsto gug^{-1}$$

Then $\text{Int}(g)$ is a smooth automorphism of $G$ and its derivative at $e$:

$$\text{Ad}(g) : T_eG \to T_eG$$

$$v \mapsto gvg^{-1}$$

is denoted $\text{Ad}(g)$. Then

$$\text{Ad} : G \to \text{GL}(\mathfrak{g})$$

is a smooth homomorphism.

Let now $G$ be connected Lie group and $K \subseteq G$ a closed subgroup.

Definition: $(G, K)$ is a riemannian symmetric pair if $\text{Ad}_G(K)$ is a compact subgroup in $\text{GL}(\mathfrak{g})$. 
(2) There is a (smooth) involution $\sigma$ of $\mathbf{F}$ with $(G^0)^0 < \kappa < G^0$.

We have seen that a RSL leads to a RSP. Conversely, the following gives a powerful way to construct RSL's.

**Thm. III. 28**

Let $(G, \kappa)$ be a RSP with involution $\sigma$. Then the homogeneous space

$$\mathcal{H} = G/\kappa$$

is a symmetric space with any $G$-invariant Riemannian metric. If $\pi : G \to \mathcal{H}$ is the quotient map, $o : = e \kappa$, and $\mathfrak{so}$ the Lie algebra symmetry at $0$, then:

$$\mathfrak{so} \circ \pi = \pi \circ \mathfrak{so}. $$
Corollary II.29. The geodesic symmetry $s$ is independent of the choice of $G$-invariant metric.

Then II.28 gives a powerful way to construct symmetric spaces.

Example II.30

(1) Let $G \subset GL(n, \mathbb{R})$ be a closed connected subgroup that is left invariant by transposition. Then 

$$\sigma : G \rightarrow G$$

$$g \mapsto g^{-1}$$

is an involutary provided (!) $G \not= O(n)$. Then $G^\sigma = G \cap O(n)$.

Actually, a non-trivial fact is that in this case $G^\sigma$ is connected, so the only possibility is $K = G^\sigma$. 
At any rate this provides us with a plethora of examples:
\( \mathrm{GL}(n, \mathbb{R}), \quad \mathrm{SL}(n, \mathbb{R}), \quad \mathrm{Sp}(2n, \mathbb{R}), \quad \mathrm{SO}(p, q) \)
with \( 1 \leq p \leq q \leq n, \quad p + q = n \).

(2) Let \( G \leq \mathrm{GL}(n, \mathbb{R}) \) be a closed connected subgroup that is invariant under \( g \mapsto g^* \) where \( g^* = \frac{t}{g} \).

Provided \( G \cap U(n), \quad \sigma(g) = \frac{t}{g}^{-1} \)
defines an involution on \( G \) and
\[ G^\sigma = G \cap U(n). \]

Again, examples are:
\( \mathrm{GL}(n, \mathbb{C}), \quad \mathrm{SL}(n, \mathbb{C}), \quad \mathrm{Sp}(2n, \mathbb{C}), \quad \mathrm{SO}(m) \times \mathbb{C} \).

(3) \( G = \mathrm{SO}(n) \), \( \text{let } r \mid_{\mathbb{R}^n} = I \),
\[ r \mid_{\mathbb{R}^n} = -I \text{.} \]
where $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$. Then

$$
\sigma(g) = r g r \quad 1 \leq p < q \leq n.
$$

is an involution on $G$ and:

$$
\mathcal{G}^\sigma = \left\{ \left( A, 0 \right) : \text{AEO}(n), \text{BEO}(q) \right\}
$$

\[ \text{det A, det B} = 1. \]

Observe that $\mathcal{G}^\sigma$ has 2 connected components, we can thus take

$K = (\mathcal{G}^\sigma)^0$ or $\mathcal{G}^\sigma$.

For example: if $p = 1$, $K = (\mathcal{G}^\sigma)^0$ we get $\text{SO}(n)/K \cong S^{n-1}$ while if $K = \mathcal{G}^\sigma$, $\text{SO}(n)/K \cong P^{n-1}(k)$.

(4) There is a similar discussion for $G = U(n)$.
(5) Comment concerning the condition that Ad \( G(K) \) rather than \( K \) is compact.
Consider the following example, to be worked out as an exercise.

Let \( G = \tilde{\text{SL}}(2, \mathbb{R}) \) be the universal covering group of \( \text{SL}(2, \mathbb{R}) \). The involution \( \sigma_1 = t_{\mathbb{R}} \) on \( \text{SL}(2, \mathbb{R}) \) lifts uniquely to an involution
\[
\overline{\sigma} : \overline{G} \to \overline{G}.
\]
Let \( \pi : \tilde{\text{SL}}(2, \mathbb{R}) \to \text{SL}(2, \mathbb{R}) \) be the canonical projection then
\[
\overline{G} = \pi^{-1}(\text{SO}(2)) \cong \mathbb{R}.
\]
Then \( \text{Ad} : \overline{G} \to \text{Ad}(\overline{G}) \)
\( \overline{G} \) is non-compact but \( \text{Ad}(\overline{G}) \cong \text{SO}(2) \setminus \{ \pm 1 \} \) is.
The proof of Thm 3.28 will consist in giving an appropriate model for \( T \otimes M \) with its \( \mathfrak{h} \)-action, to identify the \( \mathfrak{g} \)-inv. riemannian metric on \( M \) and to write down the geodesic symmetry. Thus most of the proof consists in preliminary remarks and constructions of objects of general interest.

Let \( g := \text{Lie}(\mathfrak{g}) \), \( \mathfrak{h} := \text{Lie}(\mathfrak{k}) \) and \( d\mathfrak{g} : g \to g \) the derivative of \( \mathfrak{g} \). Since \( (\mathfrak{g}^-)^0 \subset \mathfrak{k} \subset \mathfrak{g}^- \), we have

\[
\mathfrak{k} = \text{Lie}(\mathfrak{k}) = \text{Lie}(\mathfrak{g}^-) = \{ x \in \mathfrak{g}^- : \exp t x \in \mathfrak{g}^- \forall t \}\]

\[
= \{ x \in \mathfrak{g}^- : \sigma(\exp t x) = \exp t x \forall t \}
\]
and since \( \sigma (\exp t x) = \exp d \sigma (x) \)
we get
\[
A_2 = \left\{ x \in y : d \sigma (x) = x \right\}.
\]

Now \( d \sigma \) is an automorphism of the Lie algebra \( y \) and clearly
\[
(d \sigma )^2 = d \sigma (\sigma^2) = \sigma d \sigma.
\]
Thus \( d \sigma \) is diagonalizable with eigen-
values 1, -1 and we have just seen that
\[
\beta = 1 - \text{eigenspace of } d \sigma.
\]
Let \( p := -t \cdot \).

Then we have a direct sum decomposition
\[
y = \beta \oplus p. \tag{2.3.20}
\]
Next, since $\mathbf{K} \subset \mathbf{G}^0$ we have
\[ \circ \text{Int}(k) = \text{Int}(k) \circ \forall k \in \mathbf{K}. \]

Indeed, $\forall g \in \mathbf{G}:
\[ (\circ \text{Int}(k))(g) = \circ (k g k^{-1}) = k g k^{-1} \]
\[ = (\text{Int}(k) \circ g) (g). \]

As a result we obtain by taking derivative of $e$:
\[ d_0 \cdot \text{Ad}(k) = \text{Ad}(k) d_0 \circ \forall k \in \mathbf{K} \]
\[ (\text{II}.32) \]

which implies that
\[ \text{Ad}(k) < 6\mathbf{L}(g) \text{ preserves } \mathbf{M} \]
\[ (\text{III}.33) \]

Let now $M = 6/k$ and for conceptual clarity denote
\[ \theta : 6/k \rightarrow 6/k \]
\[ x \mapsto gxk \]
the diffeomorphism of $M$ given by left $g$-action.
Now we observe that \( \forall k \in \mathbb{K} \) the diagram:

\[
\begin{array}{ccc}
G & \xrightarrow{\text{Int}(k)} & G \\
\downarrow \quad \rightarrow & \quad \downarrow \pi & \downarrow \pi \\
\mathbb{C}/k & \rightarrow & \mathbb{C}/k \\
\end{array}
\]

commutes. Indeed:

\[
\text{Int}(k)(g) = \pi (kgk^{-1}) = \ell_k gk
\]

\[
= L_k (gk) = L_k \pi(g).
\]

Passing to the derivatives at \( e \) resp. \( \ell_0(e) = 0 \) we get that:

\[
\begin{array}{ccc}
\text{Ad}(k) & \quad \rightarrow \quad & \text{Ad}(k) \\
\downarrow \quad \rightarrow & \quad \downarrow \ell_k \quad \rightarrow & \downarrow \ell_k \\
\text{To}_0 \mathbb{M} & \rightarrow & \text{To}_0 \mathbb{M} \\
\end{array}
\]

commutes.

By construction of the differentiable