Proof:

~~(a)~~ Observe that since $s_0 = s_0^{-1}$,

$\sigma(g) = s_0 g s_0 = s_0 g s_0^{-1}$ is an automorphism

and

$\sigma^2(g) = s_0^2 g s_0^2 = g$.

Now $\sigma(g) = g$ ~~is equivalent to:~~

means $s_0 g s_0 = g$ which is equivalent

to $s_{g o} = g s_0 g^{-1} = s_0$.

Thus, if $g o = o$, we conclude $g s_0 g^{-1} = s_0$

hence $s_0 g s_0 = g$ that is $g \in G^\sigma$, which

shows $K \subset G^\sigma$. Finally let's show $(G^\sigma)^0 \subset K$.

Let $V \ni o$ be an open neighborhood of o

such that o is the only fixed point of s_0 in V.

Then $U := \{ g \in G^\sigma : g o \in V \}$ is

an open neighborhood of e in G^σ. If now

$g \in U$ we have $s_{g o} = s_0$ and hence $g o \in V$

is a f.p. of s_o in V implying $g_o = 0$.

Thus $\mathcal{U} \subset K$ and therefore the open subgroup of G^σ generated by $\mathcal{U}$ is contained in K implying $(G^\sigma)^\circ \subset K$.

$\boxed{\xi}$

<u>Example II.2c</u>. Work out the following example:

(1) $M = S^2$, $G = Is(M)^\circ = SO(3)$

take $o = e_3 \in S^2$; then $s_o = \begin{pmatrix} -Id & \begin{smallmatrix}0\\0\end{smallmatrix} \\ 0\ 0 & 1 \end{pmatrix}$.

$$G^\sigma = \left\{ \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} : \begin{array}{l} A \in O(2) \\ d = \pm 1 \\ \det A \cdot d = 1 \end{array} \right\}$$

has two conn. components and

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in SO(2) \right\} \text{ is}$$

connected.

Let G be a connected Lie group; $\mathfrak{g}$ its Lie algebra. For every $g \in G$, let

$$\begin{aligned} \mathrm{Int}(g) : G &\longrightarrow G \\ h &\longmapsto g h g^{-1} \end{aligned}$$

Then ~~$\mathrm{Int}(g)(e)$~~ $\mathrm{Int}(g)$ is a smooth autom. of G and its derivative at e:

$$\mathrm{Ad}(g) : \underset{\mathfrak{g}}{\underset{\shortparallel}{T_e G}} \longrightarrow \underset{\mathfrak{g}}{\underset{\shortparallel}{T_e G}}$$

is denoted $\mathrm{Ad}(g)$. Then

$$\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g})$$

is a smooth homomorphism.

Let now G be connected Lie group and $K < G$ a closed subgroup.

<u>Def. II.27</u>: (G, K) is a riemannian symm. pair if (i) $\mathrm{Ad}_G(K)$ is a compact subgroup in $GL(\mathfrak{g})$.

(2) there is a (smooth) involution σ of G

with $(G^\sigma)^\circ \subset K \subset G^\sigma$.

We have seen that a RSS leads to a RSP. Conversely, the following gives a powerful way to construct RSS's.

<u>Thm. II. 28</u>.

Let (G,K) be a RSP with involution σ. Then the homogeneous space

$$M := G/K$$

is a symmetric space wrt <u>any</u> G-invariant riemannian metric. If $\pi : G \to M$ is the quotient map, $o := eK$ and S_o the geodesic symmetry at o, then:

$$S_o \circ \pi = \pi \circ \sigma .$$

<u>Corollary II.29</u>. The geodesic symmetry s_o is independent of the choice of G-inv. metric.

Thm II.28 gives a powerful way to construct symmetric spaces.

<u>Example II.30</u>

(1) Let $G < GL(n, \mathbb{R})$ be a closed connected subgroup that is left invariant by transposition. Then

$$\sigma : G \longrightarrow G$$
$$g \longmapsto {}^t g^{-1}$$

is an involution, provided (!) $G \not\subset O(n)$.

Then $G^\sigma = G \cap O(n)$.

Actually, a non-trivial fact is that in this case G^σ is connected, so the only possibility is $K = G^\sigma$.

At any rate this provides us with a plethora of examples:

$GL_+(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $Sp(2n,\mathbb{R})$, $SO(p,q)^\circ$

with $1 \leq p \leq q \leq n$, $p+q=n$.

(2) Let $G < GL(n,\mathbb{C})$ be a closed connected subgroup that is invariant under $g \mapsto g^*$ where $g^* = {}^t\bar{g}$.

Provided $G \not\subset U(n)$, $\sigma(g) := {}^t\bar{g}^{-1}$ defines an involution on G and

$$G^\sigma = G \cap U(n).$$

Again, examples are:

$GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $Sp(2n,\mathbb{C})$, $SO(m,\mathbb{C})$.

(3) $G = SO(n)$, let $r|_{\mathbb{R}^p} = \mathrm{Id}$,

$r|_{\mathbb{R}^q} = -\mathrm{Id}$

where $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$. Then

$$\sigma(g) = rgr \qquad 1 \leq p \leq q \leq n.$$

is an involution on G and:

$$G^\sigma = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : \begin{array}{l} A \in O(p), \ B \in O(q) \\ \det A \cdot \det B = 1 \end{array} \right\}$$

Observe that G^σ has 2 connected components we can thus take

$K = (G^\sigma)^0$ or G^σ.

For example: if $p = 1$, $K = (G^\sigma)^0$ we get $SO(n)/K \simeq S^{n-1}$ while if $K = G^\sigma$, $SO(n)/K \simeq \mathbb{P}^{n-1}(\mathbb{R})$

(4) There is a similar discussion for $G = U(n)$.

(5) Comment concerning the condition that $Ad_G(K)$ rather than K is compact.

Consider the following example, to be worked out as an exercise.

Let $G = \widetilde{SL(2,\mathbb{R})}$ be the universal covering group of $SL(2,\mathbb{R})$. The involution $\sigma(g) = {}^t g^{-1}$ on $SL(2,\mathbb{R})$ lifts uniquely to an involution

$$\tilde{\sigma} : G \longrightarrow G .$$

If $\pi : \widetilde{SL(2,\mathbb{R})} \longrightarrow SL(2,\mathbb{R})$ is the canonical projection then

$$G^{\tilde{\sigma}} = \pi^{-1}(SO(2)) \cong \mathbb{R} .$$

Then ~~$Ad : G^{\tilde{\sigma}} \longrightarrow Ad(G^{\sigma}$~~

$G^{\tilde{\sigma}}$ is non-compact but $Ad(G^{\tilde{\sigma}}) \simeq SO(2)/(\pm Id)$ is.

The proof of Thm II. 28 will consist in giving an appropriate model for $T_o M$ with its K-action, to identify the G-inv. riemannian metrics on M and to write down the geodesic symmetry. Thus most of the proof consists in preliminary remarks and constructions of objects of general interest.

Let $\mathfrak{g} := Lie(G)$, $\mathfrak{k} := Lie(K)$ and $d_e\sigma : \mathfrak{g} \to \mathfrak{g}$ the derivative of σ.

Since $(G^\sigma)^\circ \subset K \subset G^\sigma$ we have

$$\mathfrak{k} = Lie((G^\sigma)^\circ) = Lie(G^\sigma)$$

$$= \{ X \in \mathfrak{g} : \exp tX \in G^\sigma \ \forall t \}$$

$$= \{ X \in \mathfrak{g} : \sigma(\exp tX) = \exp tX \ \forall t \}$$

and since $\sigma(\exp tX) = \exp(t\, d_e\sigma(X))$

we get

$$\mathfrak{h} = \{ X \in \mathfrak{g} : d_e\sigma(X) = X \}.$$

Now $d_e\sigma$ is an automorphism of the Lie Algebra $\mathfrak{g}$ and clearly

$$(d_e\sigma)^2 = d_e(\sigma^2) = Id_{\mathfrak{g}}.$$

Thus $d_e\sigma$ is diagonalizable with eigenvalues $+1, -1$ and we have just seen that

$\mathfrak{h} = +1$ – eigenspace of $d_e\sigma$.

Let $\mathfrak{p} := -1$ – " " " .

Then we have a direct sum dec.

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}. \qquad (\text{II } 31)$$

12.3.20.

Next: since $K \subset G^\sigma$ we have

$$\sigma \, \mathrm{Int}(k) = \mathrm{Int}(k)\, \sigma \quad \forall k \in K.$$

Indeed, $\forall g \in G$:

$$(\sigma \, \mathrm{Int}(k))(g) = \sigma(k g k^{-1}) = k \sigma(g) k^{-1}$$

$$= (\mathrm{Int}(k)\, \sigma)(g).$$

As a result we obtain by taking derivative at e:

$$d_e\sigma \circ \mathrm{Ad}(k) = \mathrm{Ad}(k)\, d_e\sigma \quad \forall k \in K \tag{II.32}$$

which implies that

$$\mathrm{Ad}(K) < GL(\mathfrak{g}) \text{ preserves } \mathfrak{p} \tag{II.33}.$$

Let now $M = G/K$ and for conceptual clarity denote

$$L_g : G/K \longrightarrow G/K$$
$$xK \longmapsto gxK$$

the diffeom. on M given by left g-transl.

Now we observe that $\forall k \in K$ the diagramm:

$$\begin{tikzcd} G \arrow[r, "\mathrm{Int}(k)"] \arrow[d, "\pi"'] & G \arrow[d, "\pi"] \\ G/K \arrow[r, "L_k"'] & G/K \end{tikzcd}$$

commutes. Indeed:

$$\pi\, \mathrm{Int}(k)(g) = \pi(kgk^{-1}) = kgK$$

$$= L_k(gK) = L_k \pi(g).$$

Passing to the derivatives at e resp. $\pi(e) = o$ we get that:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}(k)"] \arrow[d, "d_e\pi"'] & \mathfrak{g} \arrow[d, "d_e\pi"] \\ T_oM \arrow[r, "d_oL_k"'] & T_oM \end{tikzcd}$$

commutes.

By construction of the differentiable