structure on $M = G/K$, π is submersive and $\operatorname{Ker} d_e\pi = \mathfrak{k}$. Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ we deduce that

$$d_e\pi\big|_{\mathfrak{p}} : \mathfrak{p} \longrightarrow T_o M$$

is an isomorphism of K-modules, that is

$$\begin{tikzcd} \mathfrak{p} \arrow[r, "Ad(k)"] \arrow[d, "d_e\pi"'] & \mathfrak{p} \arrow[d, "d_e\pi"] \\ T_o(G/K) \arrow[r, "d_o L_k"'] & T_o(G/K) \end{tikzcd} \qquad (\text{II}.34)$$

commutes.

This allows us now to classify the G-invariant riemannian metrics in G/K.

Let indeed B_p, $p \in M$ be a G-invariant riemannian metric; then if $p = g \cdot o$

we have

$$B_p\left(d_o L_g(v), d_o L_g(w)\right) = B_o(v, w)$$

$$\forall v, w \in T_o M$$

which shows that B is determined by B_o and setting $g = k \in K$ shows that B_o is $D_o L_k$ – invariant $\forall\, k \in K$.

Thus G – inv. riemannian metrics corr. to $\{d_o L_k : k \in K\}$ – invariant scalar products on $T_o M$ and hence using (II. 34) to $Ad_G(K)$ – invariant scalar product on $\mathfrak{p}$.

Since $Ad_G(K) < GL(\mathfrak{g})$ is a compact subgroup there exists at least one $Ad(K)$ – invariant scalar product on $\mathfrak{g}$ and hence on $\mathfrak{p}$. Thus we have shown that there is a G-invariant riemannian metric on $M = G/K$.

Let $o = eK$ and define:

$$s_o(g \cdot o) = \sigma(g) \cdot o$$

Let's see that this is well defined:

$\forall k \in K$: $\quad \sigma(gk) \cdot o = \sigma(g)\sigma(k) \cdot o$

$$= \sigma(g)\, k \cdot o = \sigma(g) \cdot o .$$

Thus $\quad s_o \pi = \pi \circ \sigma$.

Taking derivative at $o = \pi(e)$ gives:

$$d_o s_o \cdot d_e \pi = d_e \pi \, d_e \sigma$$

which for $X \in \mathfrak{p}$ gives:

$$d_o s_o (d_e \pi(X)) = - d_e \pi (X)$$

hence $\quad d_o s_o = - Id_{T_o M}$.

Now: since $(L_k s_o L_{k^{-1}})(g\cdot o) = k\, s_o(k^{-1}g\cdot o)$

$$= k\,\sigma(k^{-1})\sigma(g).o = \sigma(g).o = s_o(g\cdot o)$$

we get that if $p = g\cdot o$,

$$s_p := L_g\, s_o\, L_{g^{-1}} \text{ is well defined.}$$

We have:

$$D_p S_p = D_o L_g\, D_o s_o\, D_{g\cdot o} L_{g^{-1}}$$

$$= -D_o L_g\, D_{g_o} L_{g^{-1}} = -Id_{T_pM}.$$

Finally we want to show that s_o is an isometry wrt any G-inv. riemannian metric. This will show the same for the s_p's and conclude the proof.

If now B_p is any G-inv. riemannian metric on M we need to verify

that
$$B_{s_0(p)}\left(D_p s_0(v), D_p s_0(w)\right) = B_p(v,w)$$
$$\forall v,w \in T_p M,$$
$$\forall p \in M.$$

To this end write $p = g\cdot o$, $v = D_o L_g(v_o)$, $w = D_o L_g(w_o)$, $v_o, w_o \in T_o M$. By G-invariance we have
$$B_p(v,w) = B_o(v_o, w_o).$$

Now we need to compute $D_p s_o$:
$$D_p s_o(v) = D_p s_o\left(D_e L_g(v_o)\right) = D_e(s_o L_g)(v_o)$$

Now:
$$(s_o L_g)(xK) = s_o(gxK) = \sigma(gx)K$$
$$= \sigma(g)\sigma(x)K = L_{\sigma(g)} s_o(xK).$$

Thus $s_o L_g = L_{\sigma(g)} s_o$ and our computation of derivatives gives:

$$D_p s_0(v) = D_0(s_0 L_g)(v_0) = D_0(L_{\sigma(g)} s_0)(v_0)$$

$$= (D_0 L_{\sigma(g)})(-v_0),$$

so $(D_p s_0)(v) = -D_0 L_{\sigma(g)}(v_0)$.

Thus:

$$B_{s_0(p)}(D_p s_0(v), D_p f(w))$$

$$= B_{\sigma(g).0}(D_0 L_{\sigma(g)}(v_0), D_0 L_{\sigma(g)}(w_0))$$

$$= B_0(v_0, W_0) = B(v, w).$$

$\boxed{\text{S}}$

<u>Remark II.35</u> Let $M \ni o$ be a riemannian symmetric space with basepoint o and (G, K) the associated riemannian symmetric pair. Recall that this means, $G = \mathrm{Is}(M)^\circ$ and $K = \mathrm{Stab}_G(o)$. Then we claim that there is a unique involution $\sigma: G \to G$ with $(G^\sigma)^\circ \subset K \subset G^\sigma$.

Let σ_1, σ_2 be involutions with

$$(G^{\sigma_i})^\circ \subset K \subset G^{\sigma_i} \qquad i = 1, 2.$$

Thm II.28 implies that if s_o is the geodesic symmetry at o and $\pi: G \to M$, $g \mapsto g.o$ the canonical projection map, then

$$s_0 \pi = \pi \sigma_1 = \pi \sigma_2 .$$

Thus $\pi \sigma_1 = \pi \sigma_2$, equivalently,

$$\sigma_1(h).o = \sigma_2(h).o \quad \forall h \in G .$$

Replacing h by gh we get

$$\sigma_1(g)\,\sigma_1(h).o = \sigma_2(g)\,\sigma_2(h).o \quad \forall g, h \in G .$$

Since G acts transitively on M let

$p \in M$ and $h \in G$ with $p = \sigma_1(h).o = \sigma_2(h).o$

Then we deduce that:

$$\sigma_1(g)p = \sigma_2(g)p \quad \forall p \in M, \forall g \in G$$

which implies $\sigma_1(g) = \sigma_2(g) \quad \forall g \in G$

hence $\sigma_1 = \sigma_2$.

Exercise II. 36

(1) Let G be a connected topological group and $N \triangleleft G$ a normal subgroup which is discrete. Show that $N \subset Z(G)$ is contained in the center $Z(G)$ of G.

(2) Let (G, K) be a riemannian symmetric pair, $Z(G)$ the center of G. Show that $Ad_G : G \longrightarrow GL(\mathfrak{g})$ induces an isomorphism of Lie groups:

$$K / _{K \cap Z(G)} \longrightarrow Ad_G(K) < GL(\mathfrak{g}).$$

Def. II. 37 Let (G, K) be a riemannian symmetric pair with involution σ. The Cartan involution is the automorphism

$$D_e\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$$

of the Lie algebra $\mathfrak{g}$ obtained by taking the derivative of σ at e. It is denoted Θ and the corresponding eigenspace decomposition (see II. 31)

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is called the Cartan decomposition with respect to Θ.

Proposition II.38

Let (G, K) be a RSP, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition w.r.t. the Cartan involution Θ and $[\ ,\ ]$ the Lie algebra bracket on $\mathfrak{g}$. Then we have:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \quad , \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \, , \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k} .$$

Proof: Let $X, Y \in \mathfrak{g}$ be eigenvectors of Θ: $\Theta(X) = \lambda X$, $\Theta(Y) = \mu Y$ with $\lambda, \mu \in \{-1, 1\}$. Since Θ is a Lie algebra automorphism we have

$$\Theta([X, Y]) = [\Theta(X), \Theta(Y)] = \lambda\mu [X, Y]$$

which implies the proposition since

$\mathfrak{k}$ is the $+1$-eigenspace and $\mathfrak{p}$ is the -1-eigenspace of Θ.