Exercise class 3, Symmetric Spaces

Thursday, 2 April 2020 08:01

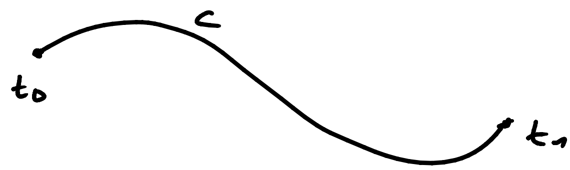
Sheet 3: Ex 2: To show: TFAE

(i) $N \subset M$ is totally geodesic

(ii) $c: \mathbb{R} \to N$, $T_{c;t_0,t_1} : T_{c(t_0)}M \longrightarrow T_{c(t_1)}M$.

Actually sends

$$J_{c;t_0,t_1}\left(T_{c(t_0)}N\right) = T_{c(t_1)}N$$



Hints: $M = \mathbb{R}^m$, $N = \mathbb{R}^n \times 0^{m-n}$ locally.

Let V be a vector field along c, then $\frac{D}{dt}V(t) = 0 \;\forall t$

$\nabla_{e_i} e_j = \sum_{k=1}^{n} \Gamma_{ij}^k e_k$, If $\gamma: \mathbb{R} \to N$ is a geodesic.

then $\nabla_{\dot\gamma}\dot\gamma = 0$.

Note: The solution can be found in

Helgason: "Differential geometry, ... symmetric spaces" 1979.
Chapter I. 14.

Ex 1: A geodesic is a curve that locally minimizes distance.

Ex 3: Goal:
$$D_X \exp = \left(\partial_e L_{\exp(X)}\right) \cdot \sum_{n=0}^{\infty} \frac{(-ad(X))^n}{(n+1)!}.$$

Review: (G,K) RSP,

$\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\pi: G \to G/K \ni o = eK \overset{\pi(e)}{=}$, $e \in G$.

$$\left\{\begin{matrix} \mathfrak{n} \subset \mathfrak{p} \text{ Lie triple system} \\ [\mathfrak{n}, [\mathfrak{n},\mathfrak{n}]] \subset \mathfrak{n} \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} N \subset M \text{ totally geodesic} \\ \text{submanifolds with} \\ o \in N \end{matrix}\right\}$$

$$\mathfrak{n} \longmapsto \mathrm{Exp}_o(\partial_e \pi(\mathfrak{n}))$$

$$(\partial_e \pi)^{-1} T_o N \longleftarrow N$$

$$\mathfrak{n} \longmapsto \mathrm{Exp}_o(d_e\pi(\mathfrak{n}))$$
$$(d_e\pi|_{\mathfrak p})^{-1}\, T_oN \longleftarrow N$$

Remark: $\cdot\ [\mathfrak p,\mathfrak p]\subset\mathfrak k$.

$$\begin{tikzcd}
\mathfrak k \arrow[loop left, "\mathrm{ad}(\mathfrak k)"] \arrow[r, bend left, "\mathrm{ad}(\mathfrak p)"] & \mathfrak p \arrow[loop right, "\mathrm{ad}(\mathfrak k)"] \arrow[l, bend left, "\mathrm{ad}(\mathfrak p)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
{[\mathfrak n,\mathfrak n]} \arrow[loop left, "\mathrm{ad}({[\mathfrak n,\mathfrak n]})"] \arrow[r, bend left, "\mathrm{ad}(\mathfrak n)"] \arrow[d, phantom, sloped, "\subset"] & \mathfrak n \arrow[loop right, "\mathrm{ad}({[\mathfrak n,\mathfrak n]})"] \arrow[l, bend left, "\mathrm{ad}(\mathfrak n)"] \arrow[d, phantom, sloped, "\subset"] \\
\mathfrak k & \mathfrak p
\end{tikzcd}$$

$$\mathfrak g = \mathfrak k \oplus \mathfrak p \qquad\qquad \boxed{\mathfrak g' = [\mathfrak n,\mathfrak n]\oplus\mathfrak n}$$

Ex: $G = SL(2,\mathbb R)$, $K = SO(2)$.

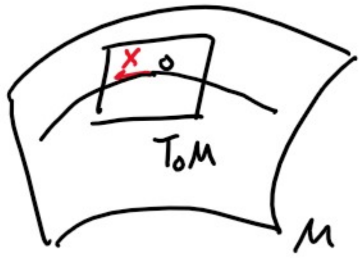
$\mathfrak{sl}(2,\mathbb R) = \{X\in\mathfrak g \mid \mathrm{tr}(X)=0\} = \mathfrak g$. $\qquad \sigma(g) = g^{t-1}$, $\quad \Theta(X) = -X^t$

$$\mathfrak g = \left\langle \begin{pmatrix}0&-1\\1&0\end{pmatrix}\right\rangle \oplus \left\langle \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right\rangle = \mathfrak k\oplus\mathfrak p. \qquad -X^t = X$$

- $\dim(\mathfrak n) = 2$: $\mathfrak p \cong T_oM$.
- $\dim(\mathfrak n) = 1$: $\mathfrak n = \langle \cancel{X}\rangle$, $X\in\mathfrak p$. $\quad [\mathfrak n,\mathfrak n] = 0$, $[\mathfrak n,[\mathfrak n,\mathfrak n]] = 0$.
- $\dim(\mathfrak n) = 0$: $0 < \mathfrak p$: Is a Lie triple system.

$$\mathrm{Exp}_o(0) = \{o\}$$

Note: $\{o\}$ is not a symmetric space.



$\cancel{s_p \neq \mathrm{Id}_M}$

Ex: $M = \underbrace{\mathfrak p}_{\mathfrak{sl}(2,\mathbb R)=\mathfrak k\oplus\mathfrak p} \times \{0\} \subset \mathfrak{gl}(3,\mathbb R)$ is a Lie triple system.

$\Rightarrow SL(3,\mathbb R)/SO(3)$ contains hyperbolic planes.

$$\mathfrak p\times\{0\} = \left\{ \begin{pmatrix} A & \begin{matrix}0\\0\end{matrix} \\ \begin{matrix}0&0\end{matrix} & 0\end{pmatrix} \in M_{3\times3}(\mathbb R) \;\middle|\; A \text{ is symmetric and } \mathrm{tr}(A) = 0 \right\}$$

Question: $d(\mathrm{Exp}X_1, \mathrm{Exp}X_2) = \dots = \exp(2(X_1-X_2))| = \|X_2 - X_1\|$

$X_1, X_2 \in \mathfrak p \subset \mathfrak{sl}(n,\mathbb R)$

The Killing form: $\quad B_{\mathfrak g} : \mathfrak g\times\mathfrak g \longrightarrow \mathbb R$

$(X, Y) \longmapsto \mathrm{tr}(\mathrm{ad}(X)\circ\mathrm{ad}(Y))$

The Killing form: $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$

$$(X, Y) \longmapsto tr(\, ad(X) \circ ad(Y)).$$

$$ad(X) : \mathfrak{g} \longrightarrow \mathfrak{g}.$$

Ex: $\mathfrak{g} = \langle \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{e_1} \rangle \oplus \langle \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{e_2}, \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{e_3} \rangle = \langle e_1, e_2, e_3 \rangle = \mathbb{R}^3$

$ad(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$,

$[e_1, e_2] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 2 \cdot e_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

$[e_1, e_3] = \ldots = -2 e_2$

$ad(e_1) e_1 = [e_1, e_1] = 0$

$[e_2, e_1] = -[e_1, e_2] = -2 e_3$

$[e_2, e_3] = \ldots = -2 e_1$

$ad(e_2) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$

$B_{\mathfrak{g}}(e_1, e_1) = tr\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix} \right) = tr \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix} = -8$

$ad(e_3) = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$\vdots$

$$B_{\mathfrak{g}}(X, Y) = X^T \begin{pmatrix} -8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} Y$$

Note: $B_{\mathfrak{g}}|_{\mathfrak{k}} << 0$, (this always happens for $\mathfrak{g} = Lie(I_s(M)^0)$)

$B_{\mathfrak{g}}|_{\mathfrak{p}} >> 0$, this is what's called: $\mathfrak{g}$ is of non-compact-type.

Sheet 2: Ex 4: (2) Let (G, K) be a RSP, Then $K/_{(K \cap Z(G))} \cong Ad_G(K)$.

$Ad : G \longrightarrow GL(\mathfrak{g})$.

Claim: $\ker(Ad) = Z(G)$

Proof: "$\supset$" Let $g \in Z(G)$, i.e. $\forall h \in G: ghg^{-1} = h$. $Int(g) h = h$.

$$Ad(g) = D_e Int(g) = D_e Id_G = Id_{\mathfrak{g}}, \Rightarrow g \in \ker(Ad).$$

"$\subset$"

$$\begin{array}{ccc} G & \xrightarrow{Int(g)} & G \\ \uparrow{\scriptstyle exp} & & \uparrow{\scriptstyle exp} \\ \mathfrak{g} & \xrightarrow[\sim]{Ad(g)} & \mathfrak{g} \end{array}$$

Let $X \in \mathfrak{g}$, then: $Int(g) \exp(X) = \exp(Ad(g) X)$.

Let $g \in \ker(Ad)$, i.e: $Ad(g) = Id_{\mathfrak{g}}$.

$\Rightarrow Int(g) \exp(X) = \exp(X)$.

$\mathfrak{g} \sim \mathfrak{g} \quad \Rightarrow \quad \underline{\mathrm{Int}(g)\exp(X) = \exp(X).}$

$\underline{\mathrm{Int}(g)\, h = h} \quad \forall\, h = \exp(X) \in$ neighborhood U of e.

$\overset{\text{Ex } 3(5)}{}$

G connected $\Rightarrow$ G is generated by any open nbhd of e.

$\forall\, k \in G: \quad k = h_1 \cdot \ldots \cdot h_n$ with h_i in nbhd U.

$\mathrm{Int}(g) k = g k g^{-1} = g h_1 g^{-1} \cdot \ldots \cdot g h_n g^{-1} = h_1 \cdot \ldots h_n = k.$ $\square$

$\Rightarrow g \in Z(G).$

$\Rightarrow \quad G/_{Z(G)} \cong \mathrm{Ad}(G). \quad \Rightarrow \quad K/_{K \cap Z(G)} \cong \mathrm{Ad}(K).$

<u>Ex:</u> let (G, K) RSP, $G \curvearrowright G/K$.

<u>Claim:</u> $Z(G) \cap K$ fixes all elements in G/K.

<u>Proof:</u> let $k \in Z(G) \cap K$, $k\, \underline{g K} = g k K = g K$ $\square$

Now if (G, K), $G = \mathrm{Is}(M)^\circ$, then $Z(G) \cap K = \{\mathrm{Id}\}$.

$\Rightarrow \quad \underline{K \cong \mathrm{Ad}(K)}.$

⚠ $\exists\, (G,K) \neq (G', K')$ RSP: $G/K = G'/K'$.

Ex: $(SL(2,\mathbb{R}), SO(2)) \neq (\widetilde{SL(2,\mathbb{R})}, \widetilde{SO}(2)) \Rightarrow \tilde{G}/\tilde{K} = G/K$ is hyp. plane.

$Z(SL(2,\mathbb{R})) \cap SO(2) = \{\pm \mathrm{Id}\}.$

$(PSL(2,\mathbb{R}), PSO(2)) = (\mathrm{Is}(\mathbb{H}^2)^\circ, \ldots)$

<u>Def:</u> If $Z(G) \cap K$ is discrete, then (G, K) is called <u>effective</u>.

(1) G connected top. $N \trianglelefteq G$ normal, discrete.

$\Rightarrow N \subset Z(G).$