Tuesday, 12 May 2020 09:59

Exercise class, 12.5.2020.

Effective / Reduced OSL.

Def: A OSL $(\mathfrak{g}, \theta)$ is *effective* if $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{k} = 0$.

Lemma: $\boxed{(\mathfrak{g}, \theta) \text{ is effective} \Leftrightarrow B_{\mathfrak{g}}|_{\mathfrak{k}\times\mathfrak{k}} << 0.}$

Proof: "$\Rightarrow$" Lemma III.6.

"$\Leftarrow$" Let $X \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{k}$, $\quad \forall Y \in \mathfrak{g}: \; [X,Y]=0 \Rightarrow ad(X)=0$

$\Rightarrow \forall Y \in \mathfrak{k}: \; B_{\mathfrak{g}}(X,Y) = tr(ad(X)\, ad(Y)) = 0.$

$\Rightarrow X = 0. \Rightarrow$ effective. $\square$

Def: OSL $(\mathfrak{g}, \theta)$ is *reduced* if $\mathfrak{k}$ contains no non-zero ideal of $\mathfrak{g}$.

Remark: Reduced $\Rightarrow$ effective, because $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{k} = 0$.

Prop: Let M be a RSS. $G = Iso(M)^\circ$, $K = Stab_G(o)$.
Then $(\mathfrak{g}, \theta)$ is reduced.

Proof: Let $\mathfrak{n} \subset \mathfrak{k}$ ideal in $\mathfrak{g}$. $\Rightarrow N \lhd G$ normal subgroup $\quad N < K$

$\forall n \in N, g \in G$ $\; ngn^{-1} \in N$

Let $gK \in G/K = M$. for some $n' \in N$

$n.gK = ng.K = gn'K \underset{N<K}{=} gK \Rightarrow n$ fixes all points in M.

$N < Iso(M) \Rightarrow N = \{id_M\} \Rightarrow \mathfrak{n} = Lie(N) = 0$ $\square$

Recall Thm IV.15: Reduced OSL's give rise to RSS.

$$\boxed{(\mathfrak{g}, \theta) \text{ is reduced} \Leftrightarrow \text{it gives rise to RSS}}$$

What to do when $(\mathfrak{g}, \theta)$ is not reduced. $\Rightarrow$ $\mathfrak{g}/_{\mathfrak{z}(\mathfrak{p})\cap\mathfrak{k}}$ is reduced.

Claim: $\mathfrak{z}(\mathfrak{p}) \cap \mathfrak{k} = \{X \in \mathfrak{k} : [X, \mathfrak{p}] = 0\}$ is an ideal in $\mathfrak{g}$.
Moreover every $\mathfrak{n} \subset \mathfrak{k}$ of $\mathfrak{g}$, then $\mathfrak{n} \subset \mathfrak{z}(\mathfrak{p}) \cap \mathfrak{k}$.

Proof: $X \in \mathfrak{z}(\mathfrak{p}) \cap \mathfrak{k}$. $\quad , \; \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

Proof: $X \in z(\mathfrak p)\cap \mathfrak k$, $\quad \mathfrak g = \mathfrak k \oplus \mathfrak p$

$$\begin{tikzcd} \mathfrak k \arrow[loop left, "ad(\mathfrak k)"] \arrow[r, bend left, "ad(\mathfrak p)"] & \mathfrak p \arrow[loop right, "ad(\mathfrak k)"] \arrow[l, bend left, "ad(\mathfrak p)"] \end{tikzcd}$$

- $[X,\mathfrak g] = \underbrace{[X,\mathfrak k]}_{\mathfrak k} + \underbrace{[X,\mathfrak p]}_{=0} \in \mathfrak k$
- $[[X,\mathfrak g],\mathfrak p] = [[X,\mathfrak k],\mathfrak p] + [\underbrace{[X,\mathfrak p]}_{=0},\mathfrak p]$

$$\subset \underbrace{[\underbrace{[\mathfrak k,\mathfrak p]}_{\subset \mathfrak p},X]}_{=0} + [\underbrace{[\mathfrak p,X]}_{=0},\mathfrak k] + 0 = 0 \Rightarrow [X,\mathfrak g]\subset z(\mathfrak p)$$

$\Rightarrow z(\mathfrak p)\cap\mathfrak k$ is an ideal in $\mathfrak g$

- Let $\mathfrak n\subset\mathfrak k$ ideal in $\mathfrak g$. $\quad [\mathfrak n,\mathfrak g]\subset\mathfrak n$

  $\underbrace{[\mathfrak n,\mathfrak p]}_{\subset\mathfrak p}\subset \mathfrak n\cap\mathfrak p\subset \mathfrak k\cap\mathfrak p = 0.$

  because $\mathfrak n\subset\mathfrak k$ □

Remark: effective $\iff z(\mathfrak g)\cap\mathfrak k = 0$ $\qquad z(\mathfrak g)$

reduced $\iff z(\mathfrak p)\cap\mathfrak k = 0$ $\qquad \cap\ z(\mathfrak p)$

Prop: $(\mathfrak g,\theta)$ is an OSL $\Rightarrow$ $\tilde{\mathfrak g} = \mathfrak g / z(\mathfrak p)\cap\mathfrak k$, $(\tilde{\mathfrak g},\tilde\theta)$ is a reduced OSL.

Proof: $z(\mathfrak p)\cap\mathfrak k$ is θ-invariant, $\theta|_{\mathfrak k} = id_{\mathfrak k}$.

$$\Rightarrow \tilde\theta : \tilde{\mathfrak g}\to\tilde{\mathfrak g}, \quad \tilde{\mathfrak g} = \mathfrak k / z(\mathfrak p)\cap\mathfrak k \oplus \mathfrak p = \tilde{\mathfrak k}\oplus\mathfrak p.$$

$$\tilde\theta : \tilde{\mathfrak k}\oplus\mathfrak p \to \tilde{\mathfrak k}\oplus\mathfrak p$$
$$X+Y \mapsto X+-Y \qquad \text{Note } \tilde\theta \neq Id.$$

$ad(\tilde{\mathfrak k}) = Lie(\tilde K)$, $\tilde K$ is compact. ($\tilde{\mathfrak k}$ compactly embedded).

$\mathfrak k$ is compactly embedded $\Rightarrow \tilde K = K/N \dots$, $z(\mathfrak p)\cap\mathfrak k = Lie(N)$

$\tilde{\mathfrak k} = \mathfrak k / z(\mathfrak p)\cap\mathfrak k$ $\qquad \Rightarrow K$ compact $\Rightarrow \tilde K$ compact.

$\qquad\Rightarrow \tilde{\mathfrak k}$ is compactly embedded.

$ad(\tilde{\mathfrak k}) = ad\left(\mathfrak k / z(\mathfrak p)\cap\mathfrak k\right)$, $\quad K < GL(\mathfrak g)$.

$\tilde K < GL(\mathfrak g / z(\mathfrak p)\cap\mathfrak k)$

to show: $(\tilde{\mathfrak g},\tilde\theta)$ reduced:

- Let $\tilde{\mathfrak n}\subset\tilde{\mathfrak k}$ an ideal in $\tilde{\mathfrak g} = \mathfrak g / z(\mathfrak p)\cap\mathfrak k$, $\Rightarrow \exists\, \mathfrak n\subset\mathfrak k$ ideal in $\mathfrak g$

  $\tilde{\mathfrak n} = \mathfrak n / z(\mathfrak p)\cap\mathfrak k$, We know $\mathfrak n\subset z(\mathfrak p)\cap\mathfrak k$ because Claim.

$\tilde{\mathfrak{r}} = \mathfrak{r} / z(\mathfrak{p}) \cap \mathfrak{k}$, We know $\mathfrak{r} \subset z(\mathfrak{p}) \cap \mathfrak{k}$ because Claim.

$\underset{\uparrow \text{largest.}}{}$

$\tilde{X} = X + z(\mathfrak{p}) \cap \mathfrak{k}$ for $X \in \mathfrak{r}$. $\Rightarrow \tilde{\mathfrak{r}} = 0. \Rightarrow (\tilde{\mathfrak{g}}, \tilde{\theta})$ reduced. $\square$

Semi-simple OSL

Def: Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}$ is semi-simple if $B_{\mathfrak{g}}$ is non-degenerate.

Recall if $(\mathfrak{g}, \theta)$ is effective OSL

$$\Rightarrow \mathfrak{g} = \underset{\text{Euclidean.}}{\mathfrak{g}_0} \oplus \underset{B_{\mathfrak{g}}|_{\mathfrak{p}\times\mathfrak{p}} >> 0}{\mathfrak{g}_+} \oplus \underset{B_{\mathfrak{g}}|_{\mathfrak{p}\times\mathfrak{p}} << 0}{\mathfrak{g}_-}$$

$\mathfrak{g}_0 = \mathfrak{p}_0 \oplus \mathfrak{k}_0$. $[\mathfrak{p}_0, \mathfrak{p}_0] = 0$ and $\mathfrak{p}_0$ ideal in $\mathfrak{g}$. "$\mathfrak{p}_0$ is a abelian ideal."

Prop: $\boxed{\mathfrak{g} \text{ is semi-simple} \iff \mathfrak{g} \text{ has no Euclidean factor.}}$

Proof: "$\Leftarrow$" $\mathfrak{g}_0 = 0 \Rightarrow \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \Rightarrow B_{\mathfrak{g}}$ non-degenerate.

"$\Rightarrow$" $\mathfrak{g}$ semi-simple. $\mathfrak{g}_0 = \mathfrak{p}_0 \oplus \mathfrak{k}_0$, $[\mathfrak{p}_0, \mathfrak{p}_0] = 0$.

$X \in \mathfrak{p}_0$

$$\mathrm{ad}(X) = \begin{array}{c} \begin{array}{ccc} \mathfrak{p}_0 & \mathfrak{k}_0 & \mathfrak{g}_\pm \end{array} \\ \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \mathfrak{p}_0 \\ \mathfrak{k}_0 \\ \mathfrak{g}_\pm \end{array}$$

Basis: $\mathfrak{g} = \mathfrak{p}_0 \oplus \mathfrak{k}_0 \oplus \mathfrak{g}_\pm$

$[X, \mathfrak{p}_0] = 0 \iff \mathrm{ad}(X)\mathfrak{p}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$

$\left.\begin{array}{l} [X, \mathfrak{k}_0] \subset \mathfrak{p}_0 \\ [X, \mathfrak{g}_\pm] \subset \mathfrak{p}_0 \end{array}\right\}$ because $\mathfrak{p}_0$ ideal in $\mathfrak{g}$.

$[Y, \mathfrak{p}_0] \subset \mathfrak{p}_0$

Let $Y \in \mathfrak{g}$

$$\mathrm{ad}(Y) = \begin{array}{c} \begin{array}{ccc} \mathfrak{p}_0 & \mathfrak{k}_0 & \mathfrak{g}_\pm \end{array} \\ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \end{array}$$

$$\Rightarrow \forall Y \in \mathfrak{g}: \ B_{\mathfrak{g}}(X, Y) = \mathrm{tr}\big(\mathrm{ad}(X)\,\mathrm{ad}(Y)\big) = \mathrm{tr}\left(\begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right) = \mathrm{tr}\begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

$\overset{B_{\mathfrak{g}} \text{ is non-deg.}}{\Rightarrow} X = 0. \Rightarrow \mathfrak{p}_0 = 0.$

If $\mathfrak{p}_0 = 0$, $\mathfrak{k}_+ = [\mathfrak{p}_+, \mathfrak{p}_+] \oplus \mathfrak{k}_0$. , $\mathfrak{g}_0 = 0$. $\Rightarrow$ No Euclidean factor. $\square$

⚠ semi-simple may still have flat subspace.

Irreducible OSL.

Def: $(\mathfrak{g}, \theta)$ is an irreducible OSL if

not just effective.

not just effective.

Def: $(\mathfrak{g},\theta)$ is an irreducible OSL if

1) $\mathfrak{g}$ is semi-simple $\wedge$ $\mathfrak{g}$ is reduced.

2) $ad(\mathfrak{k}) \curvearrowright \mathfrak{p}$ such that the only invariant subspaces are $\mathfrak{p}$ and 0.

Duality:

Let $\mathfrak{g}_0 = \mathfrak{sl}(n,\mathbb{R}) = \{X \in \mathfrak{gl}(n,\mathbb{R}) : tr(X) = 0\}$, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$

$\theta_0 : X \longrightarrow -X^T$ $\qquad X + X^T = 0$

$\mathfrak{k}_0 = E_1(\theta_0) = \{X \in \mathfrak{g} : \theta_0(X) = X\} = \mathfrak{so}(n)$ anti-symmetric.

$\mathfrak{p}_0 = E_{-1}(\theta_0) = \{X \in \mathfrak{g} : X = X^T\}$, symmetric.

$\mathfrak{g} = \mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0 + i\mathfrak{g}_0 = \{Z = X + iY : X, Y \in \mathfrak{sl}(n,\mathbb{R})\} = \mathfrak{sl}(n,\mathbb{C})$.

$= \mathfrak{k}_0 \oplus \mathfrak{p}_0 \oplus i\mathfrak{k}_0 \oplus i\mathfrak{p}_0$

$\mathfrak{g}^* = \mathfrak{k}_0 \oplus i \cdot \mathfrak{p}_0 = \{Z = X + iY \in \mathfrak{g}^{\mathbb{C}} : X \in \mathfrak{so}(n), Y^T = Y\}$

$= \{Z \in \mathfrak{g}^{\mathbb{C}} : Z + \overline{Z}^T = 0\}$ $\qquad X + \overline{X}^T = 0$, $\quad iY = iY^T$

$iY - iY^T = 0$

$iY + \overline{(iY)}^T = 0$

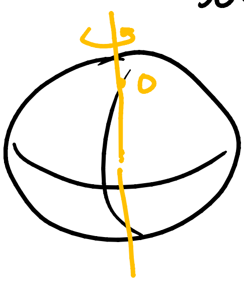
$= \mathfrak{su}(n)$

$\mathfrak{k}^* = \mathfrak{k} = \mathfrak{so}(n)$.

$\Rightarrow SU(n)/SO(n)$ is the compact dual of $SL(n,\mathbb{R})/SO(n) = P_1(n)$

$\underline{n=2}$: Fact: $SU(2) \longrightarrow SO(3)$ is a double-cover. $\mathfrak{su}(2) = \mathfrak{so}(3)$

$\Rightarrow SU(2)/SO(2) = SO(3)/SO(2) = \mathbb{S}^2$, is compact dual of $P_1(2) = \mathbb{H}^2$.



$Stab_{SO(3)}(o) = SO(2) \times \{1\}$

M RSS $\longrightarrow (G, K)$

$n \geq 3$: $SU(n)$ is not related to $SO(n-1)$..

<u>$n \geq 3$</u>: $SU(2n)$ is not related to $SO(n-1)$..