Exercise 1.1 Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$ and let $\Omega :=]a, b[\times]c, d[\subset \mathbb{R}^2$.

(i) Find all the harmonic functions on Ω that are of the form $u(x, y) = v(x)w(y)$.

You may appeal to the *unique continuation principle* stating that if $u, \tilde{u} \in C^2(\Omega)$ satisfy $\Delta u = 0 = \Delta \tilde{u}$ and $u|_Q = \tilde{u}|_Q$ for some open set $Q \subset \Omega$, then $u = \tilde{u}$ in Ω.

(ii) Prove or disprove: For every $u_0 \in C^2(\partial\Omega)$ there is a solution of the form $u(x, y) = v(x)w(y)$ to the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

Exercise 1.2 We have seen that every harmonic function satisfies the mean value property. Prove that the converse is true: let $\Omega \subset \mathbb{R}^n$ be open and let $u \in C^2(\Omega)$ satisfy the mean-value property, i. e. for any $y \in \Omega$ and any $r > 0$ such that $B_r(y) \subset \Omega$,

$$u(y) = \fint_{\partial B_r(y)} u \, d\sigma = \fint_{B_r(y)} u \, dx.$$

Prove that u is harmonic.

Exercise 1.3 (Liouville's Theorem)

(i) Suppose $u \in C^2(\mathbb{R}^n)$ is harmonic and $u \in L^1(\mathbb{R}^n)$. Prove that $u = 0$.

(ii) Suppose $u \in C^2(\mathbb{R}^n)$ is harmonic and bounded. Prove that u is constant.

Exercise 1.4 (Harnack's Inequality) Let $\Omega \subset \mathbb{R}^n$ be open. Let $Q \subset \Omega$ be any bounded and connected open subset such that $\overline{Q} \subset \Omega$. Prove that there exists a constant C depending only on Q such that for every *non-negative* harmonic function $u \in C^2(\Omega)$ there holds

$$\sup_Q u \leq C \inf_Q u.$$