Exercise 2.1 Let $\Omega \subset \mathbb{R}^n$ be open and bounded with regular boundary. Prove that Green's function G for Ω is symmetric, i.e.:

$$G(x, y) = G(y, x) \quad \forall\, x, y \in \Omega,\, x \neq y.$$

Exercise 2.2 Compute Green's function for the upper half-space $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$.

Exercise 2.3 We have computed the fundamental solution Φ of the Laplace operator in $\mathbb{R}^n$ for $n \geq 2$. When $n = 1$, the Laplacian reduces to the second derivative: $\Delta u = u''$.

(i) Arguing as for the case $n \geq 2$, compute the fundamental solution of the Laplace operator on $\mathbb{R}$. Check that the solution you find is consistent with the fundamental theorem of calculus:

$$u''(x) = f(x) \quad \Longleftrightarrow \quad u(x) = ax + b + \int_{-\infty}^{x} \int_{-\infty}^{y} f(z)dzdy,$$

for $f \in C^2_c(\mathbb{R})$, where $a, b \in \mathbb{R}$.

(ii) Compute Green's function for the one-dimensional Dirichlet problem on an interval $(a, b) \subseteq \mathbb{R}$.

Remark. In solving (i), to determine Φ uniquely remember that we also require the fundamental solution to be homogeneous.

Exercise 2.4 (**Ladyženskaja's inequality**) Prove that for any $u \in C^\infty_c(\mathbb{R}^2)$ there holds:

$$\int_{\mathbb{R}^2} |u|^4 dx \leq C\left(\int_{\mathbb{R}^2} |u|^2 dx\right)\left(\int_{\mathbb{R}^2} |\nabla u|^2 dx\right),$$

where C is a constant independent of u.

Remark. This inequality is a first example of a family of important interpolation inequalities known as Gagliardo-Nirenberg inequalities.

Exercise 2.5 Let $\Omega \subset \mathbb{R}^n$ be open, bounded and regular, $2 \leq p < \infty$ and $g \in C^2(\bar{\Omega})$. Consider

$$E_p(u) := \int_\Omega |\nabla u|^p \, dx, \qquad \mathfrak{A} := \{u \in C^2(\overline{\Omega}) \mid u|_{\partial\Omega} = g\}.$$

(i) Show that there is at most one function $u \in \mathfrak{A}$ satisfying

$$E_p(u) = \inf_{v \in \mathfrak{A}} E_p(v).$$

(ii) Find the equation satisfied by minimizers u of E_p as in (i).

(iii) Prove that for every $u \in C^2(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$ the inequality

$$\int_\Omega |\nabla u|^p \, dx \leq C_{p,n} \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |D^2 u|^p \, dx \right)^{\frac{1}{2}},$$

holds with a constant $C_{p,n}$ depending only on p and n.

Hints to Exercises.

2.1 Show that for every $\varphi, \psi \in C_c^\infty(\Omega)$ there holds

$$\int_\Omega \int_\Omega G(x,y)\varphi(y)\psi(x)\,dx\,dy = \int_\Omega \int_\Omega G(y,x)\varphi(y)\psi(x)\,dx\,dy.$$

2.2 Argue by suitably reflecting $\Phi(x-y)$ along the boundary $\partial\mathbb{R}^n_+$.

2.4 Fix first $x_1 \in \mathbb{R}$ and apply and apply the fundamental theorem of calculus to $u^2(x_1,\cdot)$, then estimate the result. Then do the same with respect to the second variable and reach the conclusion using Fubini and Cauchy-Schwarz.

2.5 For (i), use that the map $\mathbb{R}^n \ni x \mapsto |x|^p$ is strictly convex. For (ii), let $\varphi \in C_c^2(\overline{\Omega})$ and consider the function $t \mapsto E_p(u + t\varphi)$.