Exercise 3.1 Let $I :=]a, b[$ for $-\infty \leq a < b \leq \infty$. Let $u \in L^p(I)$ and let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in the Sobolev space $W^{1,p}(I)$ with $\|u - u_k\|_{L^p(I)} \to 0$ as $k \to \infty$.

(i) If $1 < p \leq \infty$, prove $u \in W^{1,p}(I)$.

(ii) Is the assumption $p \neq 1$ in part (i) necessary?

Exercise 3.2 Consider the function $f(x) = \log|x|$. From one variable calculus we know that $f \in L^p((-1, 1))$ for every $p \in [1, \infty)$.

(i) Prove that f does not have a weak derivative in any $L^p((-1, 1))$.

(ii) Prove that instead there holds, for every $\varphi \in C_c^\infty((-1, 1))$,

$$-\int_{-1}^{1} f(x)\varphi'(x)dx = \mathrm{p.\,v.}\int_{-1}^{1} \frac{\varphi(x)}{x}dx := \lim_{\varepsilon\to 0}\int_{(-1,1)\setminus[-\varepsilon,\varepsilon]} \frac{\varphi(x)}{x}dx,$$

The operator $\varphi \mapsto \mathrm{p.\,v.}\int_{-1}^{1} \frac{\varphi(x)}{x}dx$ is called *Cauchy principal value* of $1/x$.

(iii) Find an explicit expression for $\mathrm{p.\,v.}\int_{-1}^{1} \frac{\varphi(x)}{x}dx$ an absolutely convergent integral involving φ.

Remark. This exercise hints at the following fact: the weak derivative of $\log|x|$ is $\mathrm{p.\,v.}(1/x)$, which is not an ordinary function of x but rather a linear operator over the space of test functions. This heuristic consideration, familiar to every physicists, becomes rigorous and systematic in the *theory of distributions*.

Exercise 3.3 Let $1 \leq p \leq \infty$. Recall from the lecture that a continuous linear extension operator $E\colon W^{1,p}(\mathbb{R}_+) \to W^{1,p}(\mathbb{R})$ can be constructed by even reflection across 0 (Satz 7.3.3).

Construct a linear operator $E\colon W^{2,p}(\mathbb{R}_+) \to W^{2,p}(\mathbb{R})$ satisfying:

- $(Eu)|_{\mathbb{R}_+} = u$ for every $u \in W^{2,p}(\mathbb{R}_+)$;
- $\|Eu\|_{W^{2,p}(\mathbb{R})} \leq C\|u\|_{W^{2,p}(\mathbb{R}_+)}$ for a constant $C > 0$ independent of u.

Exercise 3.4 *Note:* this exercise is supplementary to the previous one. Solve that first! The solution you find may differ from the procedure here described.

(i) Let $k \in \mathbb{N}$. Show that there exist $a_1, \ldots, a_k \in \mathbb{R}$ such that for any polynomial $p\colon \mathbb{R} \to \mathbb{R}$, $p(x) = \sum_{\ell=0}^{k-1} p_\ell x^\ell$ of degree $k - 1$ and every $x < 0$, there holds

$$\sum_{j=1}^{k} a_j\, p\Big(\frac{-x}{j}\Big) = p(x).$$

(ii) Let $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. Let $a_1, \dots, a_k \in \mathbb{R}$ as in (i). Prove that the map

$$E\colon u \mapsto Eu, \qquad (Eu)(x) := \begin{cases} u(x) & \text{for } x > 0, \\ \displaystyle\sum_{j=1}^{k} a_j u\Big(\frac{-x}{j}\Big) & \text{for } x < 0, \end{cases}$$

defines a linear operator $E\colon W^{k,p}(\mathbb{R}_+) \to W^{k,p}(\mathbb{R})$ so that for every $u \in W^{k,p}(\mathbb{R}_+)$ and any integer $0 \leq \alpha \leq k$

$$\|D^\alpha(Eu)\|_{L^p(\mathbb{R})} \leq C\|D^\alpha u\|_{L^p(\mathbb{R}_+)},$$

for a constant $C > 0$ independent of u.

Hints to Exercises.

3.1 Recall that L^p spaces are reflexive for $p \in (1, \infty)$ and that L^∞ is the dual of the separable space L^1.

3.2 write $\int_{-1}^{1} f(x)\varphi'(x)dx = \lim_{\varepsilon \to 0} \int_{(-1,1)\setminus[-\varepsilon,\varepsilon]} f(x)\varphi'(x)dx$, use that f is smooth away from 0 and integrate by parts.

3.3 Argue carefully by odd reflection, and then use cut-off functions.