Exercise 4.1 Let $I :=]a, b[$ for $-\infty < a < b < \infty$. Given $f \in C^0(\overline{I})$, consider the boundary value problem

$$\begin{cases} -v'' + v = f & \text{in } I, \\ v(a) = 0, \\ v(b) = 0, \end{cases} \tag{$*$}$$

(i) Show that $(*)$ has a weak solution $u \in H_0^1(I)$, that is, satisfying

$$\int_I u'\varphi'\,dx + \int_I u\varphi\,dx = \int_I f\varphi\,dx$$

for every $\varphi \in H_0^1(I)$, and that it is unique.

(ii) Prove that the weak solution u from (i) is in fact a classical solution $u \in C^2(\overline{I})$.

(iii) Prove that the boundary-value problem

$$\begin{cases} -v'' + v = g & \text{in } I, \\ v(a) = \alpha, \\ v(b) = \beta, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$ and $g \in C^0(\overline{I})$ has unique classical solution $v \in C^2(\overline{I})$.

Exercise 4.2 Let $I :=]a, b[$ for $-\infty < a < b < \infty$. Let $g \in C^1(\overline{I})$ and $h, f \in C^0(\overline{I})$. Assume that $g(x) \geq \lambda > 0$ and $h(x) \geq 0$ for every $x \in \overline{I}$ and consider the boundary value problem

$$\begin{cases} -(g\,u')' + h\,u = f & \text{in } I, \\ u(a) = 0, \\ u(b) = 0. \end{cases} \tag{$\dagger$}$$

(i) Apply the Riesz representation theorem in a suitable Hilbert space to prove that $(\dagger)$ has a unique weak solution $u \in H_0^1(I)$.

(ii) Prove that the weak solution u from (i) is in fact a classical solution $u \in C^2(\overline{I})$.

Exercise 4.3 Let $1 \leq p \leq \infty$, $I =]a, b[$ for $-\infty < a < b < \infty$ and $u \in W^{1,p}(I)$.

(i) Let $G \in C^1(\mathbb{R})$. Prove that $G \circ u$ is in $W^{1,p}(I)$ and that the chain rule holds for weak derivatives:

$$(G \circ u)' = (G' \circ u)u'.$$

(ii) Prove that $|u| \in W^{1,p}(I)$ and compute its weak derivative.

Exercise 4.4 (Euler's Paradox) Consider the problem of minimizing the functional

$$\mathscr{F}(u) = \int_0^1 (u'(t)^2 - 1)^2 dt$$

among functions $u : [0,1] \to \mathbb{R}$ subject to the boundary condition:

$$u(0) = u(1) = 0.$$

(i) Prove that every $u \in C^1([0,1])$ solving the above minimization problem must solve the boundary value problem

$$\begin{cases} u'(t)(u'(t)^2 - 1) = c & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{$\bowtie$}$$

for some $c \in \mathbb{R}$.

(ii) Find a classical solution to ($\bowtie$) for a c of your choice. Is it also a solution to the minimization problem for $\mathscr{F}$?

(iii) Find a weak solution $u \in W^{1,\infty}((0,1))$ solving both ($\bowtie$) and the minimization problem. Is this solution unique?

(iv) Compute the value

$$\inf \left\{ \mathscr{F}(u) : u \in C^1((0,1)), u(0) = 0 = u(1) \right\}.$$

Is this value attained by some $u \in C^1((0,1))$?

Hints to Exercises.

4.1 For (i), use the Riesz representation theorem. For (iii), construct first directly a function satisfying the boundary conditions and then resort to (i)-(ii).

4.2 Apply Riesz to a suitable Hilbert space, and argue similarly as in 4.1.

4.3 Argue by approximation on u for (i) and on $x \mapsto |x|$ for (ii).

4.4 For (ii), notice that $\mathscr{F}(u) \geq 0$, and $\mathscr{F}(u) = 0$ if and only if...