Exercise 5.1 For $p \in [1, \infty]$, the space of *periodic Sobolev functions* $W^{1,p}_{\text{per}}((0,2\pi))$ is the subset of functions $\varphi \in W^{1,p}((0,2\pi))$ so that

$$\int_0^{2\pi} \varphi'(x)\psi(x)dx = -\int_0^{2\pi} \varphi(x)\psi'(x)dx$$

for every smooth function $\psi \in C^\infty([0,2\pi])$ so that $\psi^{(k)}(0) = \psi^{(k)}(2\pi)$ for every $k \in \mathbb{N}$. A similar definition is given for $W^{k,p}_{\text{per}}((0,2\pi))$.

Recall that, for a periodic function $\varphi : (0,2\pi) \to \mathbb{R}$, its Fourier coefficients are

$$\widehat{\varphi}(n) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(x)\mathrm{e}^{-inx}dx, \quad n \in \mathbb{N},$$

and its Fourier series is

$$FS(\varphi)(x) = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(n)\mathrm{e}^{inx}, \quad x \in [0,2\pi].$$

Fact: For $\varphi \in L^2((0,2\pi))$, $FS(\varphi)$ converges to φ in L^2-norm.

(i) Prove that $\varphi \in W^{1,2}_{\text{per}}((0,2\pi))$ if and only if

$$\sum_{n\in\mathbb{Z}} (1+n^2)|\widehat{\varphi}(n)|^2 < \infty,$$

(ii) Prove Sobolev embedding for periodic functions using only (i) and the "Fact" above, that is, show that if $\varphi \in W^{1,2}_{\text{per}}((0,2\pi))$, then φ can be identified with a function in $C^0[0,2\pi]$ so that $\varphi(0) = \varphi(2\pi)$ and

$$\|\varphi\|_{C^0((0,2\pi))} \le C\|\varphi\|_{W^{1,2}((0,2\pi))}.$$

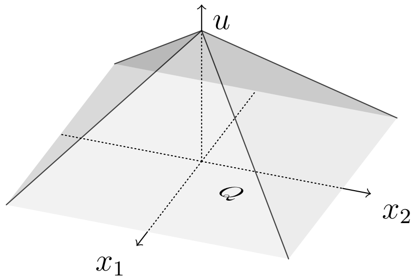
(iii) Argue similarly as in (i) and prove functions $\varphi \in W^{k,2}_{\text{per}}((0,2\pi))$ are exactly those so that

$$\sum_{n\in\mathbb{Z}} (1+n^{2k})|\widehat{\varphi}(n)|^2 < \infty.$$

Exercise 5.2 Find an open set $\Omega \subset \mathbb{R}^2$ and a function $u \in W^{1,\infty}(\Omega)$ which is not Lipschitz continuous.

Exercise 5.3 (A tent for Rudolf L.) Let $Q = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| < 1,\ |x_2| < 1\}$. Let $u\colon Q \to \mathbb{R}$ be given by

$$u(x_1, x_2) = \begin{cases} 1 - x_1, & \text{if } x_1 > 0 \text{ and } |x_2| < x_1, \\ 1 + x_1, & \text{if } x_1 < 0 \text{ and } |x_2| < -x_1, \\ 1 - x_2, & \text{if } x_2 > 0 \text{ and } |x_1| < x_2, \\ 1 + x_2, & \text{if } x_2 < 0 \text{ and } |x_1| < -x_2. \end{cases}$$



For which exponents $1 \leq p \leq \infty$ is $u \in W^{1,p}(Q)$?

Exercise 5.4 Let $\Omega \subset \mathbb{R}^n$ be open. Given $1 \leq p < \infty$, let $u \in W^{1,p}(\Omega)$.

(i) Let $u_+(x) = \max\{u(x), 0\}$ and $u_-(x) = -\min\{u(x), 0\}$. Prove $u_+, u_- \in W^{1,p}(\Omega)$ and show that their weak gradients are given by

$$\nabla u_+(x) = \begin{cases} \nabla u(x) & \text{for almost all } x \text{ with } u(x) > 0, \\ 0 & \text{for almost all } x \text{ with } u(x) \leq 0, \end{cases}$$

$$\nabla u_-(x) = \begin{cases} -\nabla u(x) & \text{for almost all } x \text{ with } u(x) < 0, \\ 0 & \text{for almost all } x \text{ with } u(x) \geq 0. \end{cases}$$

(ii) Given $u, v \in W^{1,p}(\Omega)$ and $w(x) = \max\{u(x), v(x)\}$ show that $w \in W^{1,p}(\Omega)$.

(iii) Prove that $\nabla u(x) = 0$ for almost all $x \in \Omega$ with $u(x) = 0$, which means that if $Z = \{x \in \Omega \mid u(x) = 0\}$ and $W = \{x \in \Omega \mid \nabla u(x) = 0 \text{ classically}\}$, then $Z \setminus W$ has Lebesgue measure zero.

(iv) Let $\lambda \in \mathbb{R}$. Conclude that $\nabla u(x) = 0$ for almost all $x \in \Omega$ with $u(x) = \lambda$.

Exercise 5.5 Let $\alpha \geq 0$. For any $\delta > 0$ we define

$$\mathscr{H}^\alpha_\delta(A) := \inf\Big\{\sum_{i=1}^\infty r_i^\alpha \mid A \subset \bigcup_{i=1}^\infty B_{r_i}(x_i),\ 0 < r_i < \delta,\ x_i \in \mathbb{R}^n\Big\}.$$

The *α-dimensional Hausdorff measure* of any subset $A \subseteq \mathbb{R}^n$ is defined by

$$\mathscr{H}^\alpha(A) := \lim_{\delta \searrow 0} \mathscr{H}^\alpha_\delta(A)$$

Suppose, $K \subset \mathbb{R}^n$ is a compact subset with $\mathscr{H}^{n-\alpha}(K) = 0$ for some $1 \leq \alpha < n$.

(i) For all $1 \leq p \leq \alpha$, prove that K has vanishing $W^{1,p}$-capacity.

(ii) Let $1 \leq p \leq q \leq \infty$ and $\frac{1}{q} + \frac{1}{\alpha} \leq 1$. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u \in L^q(\Omega) \cap C^1(\Omega \setminus K)$ with $|\nabla u| \in L^p(\Omega \setminus K)$. Prove that $u \in W^{1,p}(\Omega)$.

Hints to Exercises.

5.1 Use Parseval's identity: $\|\sum_n a_n e^{inx}\|^2_{L^2(0,2\pi)} = \frac{1}{2\pi}\sum_n |a_n|^2$ and the Cauchy-Schwarz inequality in the space of ℓ^2 sequences.

5.2 The domain must be nonconvex.

5.3 Who is Rudolf L.?

5.4 For (i), consider the function $G_\varepsilon \circ u$, where $G_\varepsilon \in C^1(\mathbb{R})$ is given by

$$G_\varepsilon(y) = \begin{cases} \sqrt{y^2+\varepsilon^2} - \varepsilon & \text{for } y \geq 0, \\ 0 & \text{for } y < 0. \end{cases}$$

5.5 Use that for any $r > 0$ there exists some $\psi \in C_c^\infty(B_{3r})$ satisfying $\psi = 1$ in B_{2r} and $|\nabla\psi| \leq \frac{2}{r}$.