Exercise 7.1 Let $1 \leq p \leq \infty$. Consider the open set

$$\Omega = (-1,1) \times (-1,1) \setminus \Big([0,1) \times \{0\}\Big) \subset \mathbb{R}^2.$$

Prove that there is no extension operator $E \colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$.

Exercise 7.2 In this exercise we want to prove that, for every bounded, C^1 domain $\Omega \subset \mathbb{R}^n$ and every $1 \leq p < \infty$, $W^{1,p}_0(\Omega)$ consists *exactly* of those functions in $W^{1,p}(\Omega)$ with vanishing trace, similarly to Remark 7.5.1 in the 1-dimensional case or Corollary 8.4.3 for the case $p = 2$.

Let $u \in W^{1,p}(\Omega)$.

(i) Prove that for every $\varphi \in C^\infty_c(\mathbb{R}^n)$ and every $i = 1, \ldots, n$ there holds

$$\int_\Omega \partial_i u \, \varphi \, dx = -\int_\Omega u \, \partial_i \varphi \, dx + \int_{\partial\Omega} u|_{\partial\Omega} \, \varphi \, \nu^i \, d\sigma,$$

where $\nu = (\nu^1, \ldots, \nu^n)$ denotes the outer unit normal of $\partial\Omega$ and $u|_{\partial\Omega} \in L^p(\partial\Omega)$ denotes the trace of u.

(ii) Consider the extension of U by zero to $\mathbb{R}^n$:

$$U(x) = \begin{cases} u(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Prove that, if the weak derivative of U exist, they are necessarily given by

$$\partial_i U(x) = \begin{cases} \partial_i u(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \Omega \end{cases} \tag{$*$}$$

for $i = 1, \ldots, n$.

Prove then that $u|_{\partial\Omega} = 0$ if and only if U is in $W^{1,p}(\mathbb{R}^n)$.

(iii) Prove that, for every $v \in W^{1,p}(\mathbb{R}^n)$ so that $v|_{\mathbb{R}^n \setminus \Omega} = 0$ then $v|_\Omega \in W^{1,p}_0(\Omega)$ and conclude.

Exercise 7.3 Show that the assumption that Ω is of class C^1 cannot be dropped in the characterization of $W^{1,p}_0(\Omega)$ given in Exercise 7.2: find a bounded, connected, open set $\Omega \subset \mathbb{R}^2$ and $w \in H^1(\mathbb{R}^2)$ satisfying $w(x) = 0$ for almost every $x \in \mathbb{R}^2 \setminus \Omega$ such that $w|_\Omega \notin H^1_0(\Omega)$.

Exercise 7.4 (Hardy's inequalities)

(i) Let $1 < p < \infty$, let $f \in L^p((0,\infty))$ and define

$$g(x) = \frac{1}{x}\int_0^x f(y)dy, \quad \text{for } x > 0.$$

Prove that $g \in L^p((0,\infty))$ with

$$\|g\|_{L^p((0,\infty))} \leq C\|f\|_{L^p((0,\infty))},$$

for some constant $C > 0$ depending only on p.

(ii) Let $n \geq 2$, $1 < p < n$, $\Omega \subseteq \mathbb{R}^n$ be an open subset and let $u \in W_0^{1,p}(\Omega)$. Then the function $x \mapsto \frac{u(x)}{|x|}$ is in $L^p(\Omega)$ with

$$\left\|\frac{u}{|\cdot|}\right\|_{L^p(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)},$$

for a constant $C > 0$ depending only on n and p.

Hints to Exercises.

7.1 Recall Exercise 5.2.

7.2 For (iii), deal first with the basic case on cylinders.

7.3 Compare with Exercises 5.2 and 7.1.

7.4 Minkowski inequality for integrals: $\| \int f(x,\cdot)dx\|_{L^p} \leq \int \|f(x,\cdot)\|_{L^p} dx$ will be useful.

For (ii), argue first for $u \in C_c^\infty(\mathbb{R}^n)$ and write u as integral of its radial derivative $u(x) = -\int_{|x|}^\infty \partial_r u(r\theta_x) dr$.