Exercise 9.1 Let $\Omega \subseteq \mathbb{R}^n$ be open, $1 \leq p < \infty$ and $\lambda > 0$. Recall that the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ is the subset of $L^p(\Omega)$ consisting of functions whose norm

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|u\|_{L^p(\Omega)} + [u]_{\mathcal{L}^{p,\lambda}},$$

is finite, where

$$[u]_{\mathcal{L}^{p,\lambda}(\Omega)} = \sup_{\substack{x_0\in\Omega,\\ 0<r<r_0}} r^{-\frac{\lambda}{p}} \|u - u_{x_0,r}\|_{L^p(\Omega\cap B_r(x_0))},$$

with $r_0 = \min\{1, \operatorname{diam}(\Omega)\}$ and $u_{x_0,r} = \frac{1}{|\Omega\cap B_r(x_0)|}\int_{\Omega\cap B_r(x_0)} u\,dx$.

(i) Prove that $\mathcal{L}^{p,\lambda}(\Omega)$ is Banach.

(ii) Let now $r_0' > 0$ be fixed and suppose that, in the definition above, r_0 is replaced by r_0'. Prove that the corresponding Campanato norm $\|\cdot\|'_{\mathcal{L}^{p,\lambda}(\Omega)}$ is equivalent to the original one, namely that there is a constant $C > 0$ so that

$$\frac{1}{C}\|\cdot\|'_{\mathcal{L}^{p,\lambda}(\Omega)} \leq \|\cdot\|_{\mathcal{L}^{p,\lambda}(\Omega)} \leq C\|\cdot\|'_{\mathcal{L}^{p,\lambda}(\Omega)},$$

where C depends only on r_0', λ, p.

Exercise 9.2 Let $1 \leq p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be open, connected and bounded of class C^1. Suppose $u \in W^{1,p}(\Omega)$ is so that

$$\mathcal{L}^n(\{x \in \Omega \mid u(x) = 0\}) \geq \alpha > 0,$$

where $\mathcal{L}^n$ denotes the Lebesgue measue on $\mathbb{R}^n$. Prove that there holds

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)},$$

for some constant $C = C(p, \alpha, n, \Omega) > 0$ independent of u.

Exercise 9.3 Given $k \in \mathbb{N}$, let $\Omega_k = Q_+ \cup A_k \cup Q_- \subset \mathbb{R}^2$, where

$$Q_+ =]1,3[\times]-1,1[,$$

$$A_k = [-1,1] \times]-\tfrac{1}{k}, \tfrac{1}{k}[,$$

$$Q_- =]-3,-1[\times]-1,1[.$$

Denote $u_{\Omega_k} = \fint_{\Omega_k} u\,dx$ and let $C_k = C(\Omega_k, p) \in \mathbb{R}$ be the best constant in the Poincaré inequality:

$$\int_{\Omega_k} |u - u_{\Omega_k}|^p\,dx \leq C(\Omega_k)\int_{\Omega_k} |\nabla u|^p\,dx$$

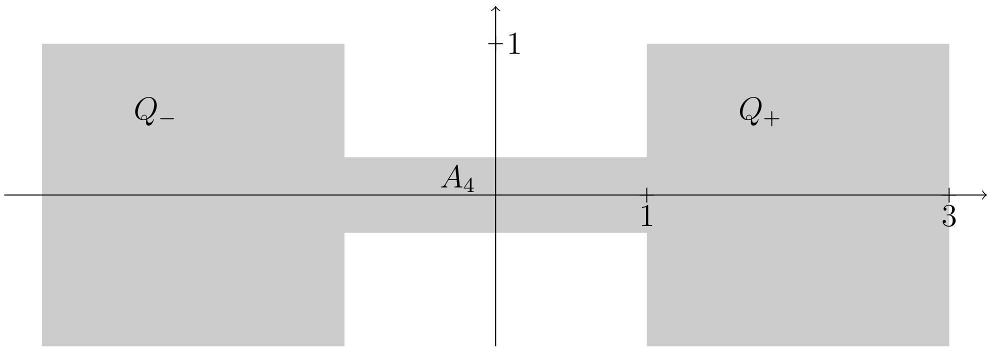

Figure 1: The domain Ω_k for $k = 4$.

for every $u \in W^{1,p}(\Omega_k)$. Prove that $C(\Omega_k) \to \infty$ as $k \to \infty$.

Exercise 9.4 Let Ω be $\mathbb{R}^n$ or a bounded domain with regular boundary. Suppose you know the validity of the inclusions:

$$W^{1,p}(\Omega) \subset L^{p*}(\Omega) \quad (1 \le p < n)$$
$$W^{1,p}(\Omega) \subset C^{0,\alpha}(\overline{\Omega}) \quad (n < p < \infty),$$

where $p^* = \frac{np}{n-p}$, $\alpha = 1 - \frac{n}{p}$, but *only* in a set-theoretic sense, without information on the topologies (for the second case, we adopt the usual convention i.e. we suppose a unique continuous representative has been identified).

Prove that this is enough to conclude that such inclusions are in fact continuous embeddings.

Exercise 9.5 Let $n < p < \infty$.

(i) Prove that for any $u \in W^{1,p}(\mathbb{R}^n)$, there holds (for its continuous representative)

$$\lim_{x\to\infty} u(x) = 0.$$

(ii) It is possible to quantify the decay of u at infinity, namely, is it possible find some $\beta = \beta(n,p) > 0$ so that

$$\text{“}u(x) = O\left(\frac{1}{|x|^\beta}\right) \quad \text{as } x \to \infty\text{”?}$$

(iii) How does the answer to (ii) change if we additionally suppose $u \in W^{k,p}(\mathbb{R}^n)$ for $k = 2, 3, \ldots$?

Hints to Exercises.

9.2 Argue by contradiction similarly as in the proof of the Poincaré inequality for $W_0^{1,p}$.

9.4 Use a suitable theorem of Functional Analysis I.

9.5 For (i), argue by approximation and use Sobolev embedding.

For (ii), construct suitably a function consisting of infinitely many smooth "bumps" that get smaller and smaller as you approach infinity...