Exercise 10.1

(i) Prove that for every $\alpha \in (0,1)$ we have the embedding $W^{2,n}(\mathbb{R}^n) \hookrightarrow C^{0,\alpha}(\mathbb{R}^n)$.

(ii) Prove that, in general, a function in $W^{2,n}(\mathbb{R}^n)$ needs *not* to be Lipschitz continuous.

Exercise 10.2

(i) Prove that there is a continuous embedding

$$W^{n,1}(\mathbb{R}^n) \hookrightarrow L^{\infty}(\mathbb{R}^n).$$

(ii) Are functions in $W^{n,1}(\mathbb{R}^n)$ also cointinuous (i.e. have a continuous representative)?

Exercise 10.3 Recall that an *algebra* is a vector space V endowed with a bilinear operation "$\cdot \times \cdot$" $: S \times S \to S$.

Let $\Omega \subseteq \mathbb{R}^n$ be a regular domain.

(i) Prove that whenever $p > n$, $W^{1,p}(\Omega)$ is an algebra with respect to the usual multiplication between functions:

$$W^{1,p}(\Omega) \times W^{1,p}(\Omega) \to W^{1,p}(\Omega), \quad f \times g = fg,$$

and moreover that such multiplication is continuous.

(ii) Prove that the statement is false when $p = n$. What is the biggest subspace $X \subset W^{1,n}(\mathbb{R}^n)$ that comes to your mind, where the multiplication becomes an algebra? Try also to find a norm for this space where the multiplication is continuous.

Exercise 10.4 (Caccioppoli's Inequality) Let $\Omega \subseteq \mathbb{R}^n$ be a domain and let $u \in W^{1,2}(\Omega)$ be a weak solution to Laplace's equation,

$$-\Delta u = 0 \quad \text{in } \Omega,$$

namely

$$\int_{\Omega} \langle \nabla u, \nabla \varphi \rangle \, dx = 0 \quad \forall \varphi \in W_0^{1,2}(\Omega). \tag{$\triangle$}$$

(i) Let $x_0 \in \Omega$ and let $R > 0$ so that $B_R(x_0) \subset \Omega$. Chose suitably a test function φ in $(\triangle)$ to prove that, for every $\rho \in (0, R)$ and every $\lambda \in \mathbb{R}$ the following inequality holds:

$$\int_{B_\rho(x_0)} |\nabla u|^2 \, dx \leq \frac{C}{(R-\rho)^2} \int_{B_R(x_0)} |u - \lambda|^2 \, dx,$$

where $C > 0$ is a universal constant.

(ii) Find the minimum of the function

$$f(\lambda) = \int_{B_R(x_0)} |u - \lambda|^2 \, dx, \quad \lambda \in \mathbb{R}.$$

Exercise 10.5 Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary.

(i) Prove that

$$[u, v] = \int_\Omega \Delta u \Delta v \, dx$$

defines a scalar product on $H^2_0(\Omega)$ equivalent to the standard one $(\cdot, \cdot)_{H^2(\Omega)}$.

(ii) Prove that for every $f \in L^2(\Omega)$ there is a unique $u \in H^2_0(\Omega)$ satisfying

$$\int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx \quad \text{for every } v \in H^2_0(\Omega)$$

(iii) Let $f \in L^2(\Omega)$. Find the boundary value problem for which u found in (ii) is a weak solution and prove that it is the unique weak solution of class H^2 for such problem.

Hints to Exercises.

10.1 For (ii) look for a counterexample involving the logarithm, similarly as those presented in the lectures dealing with capacity (§8.1)

10.2 Modify the proof of Satz 8.5.1.

10.4 For (i), look for φ equal to u times a suitable cutoff function for $B_\rho(x_0)$. Then work your way with Hölder, Young and the properties of the cutoff you have chosen.

For (ii), look at the derivative of f.