Exercise 11.1 What follows is often useful in regularity theory.

Let $p \in [1, \infty)$.

(i) Let $\Omega \subset \mathbb{R}^n$ be a domain with finite and nonzero Lebesgue measure. Prove that there exists a constant $C = C(p, \Omega) > 0$ depending only on p so that for every $u \in L^p(\Omega)$ there holds

$$\int_\Omega |u - u_\Omega|^p \, dx \leq C \inf_{\lambda \in \mathbb{R}} \int_\Omega |u - \lambda|^p \, dx,$$

where $u_\Omega = \fint_\Omega u \, dx$ is the average of u over Ω.

(ii) Let $\Omega \subseteq \mathbb{R}^n$ be a domain, $\lambda > 0$ and $u \in L^p_{\text{loc}}(\Omega)$. an let Ω', Ω'' be bounded domains with Ω' of Type A and

$$\Omega' \subset\subset \Omega'' \subset\subset \Omega.$$

Fix and $R > 0$ so that $B_R(x_0) \subset \Omega''$ for every $x_0 \in \Omega'$. Suppose you know that

$$\{u\}^p_{\mathcal{L}^{p,\lambda}(\Omega')} := \sup_{\substack{x_0 \in \Omega' \\ \rho \in (0, r_0)}} \frac{1}{\rho^\lambda} \int_{B_\rho(x_0)} |u(x) - u_{B_\rho(x_0)}|^p \, dx < \infty.$$

Use (i) to prove that then u belongs to $\mathcal{L}^{p,\lambda}(\Omega')$ (the usual Campanato space) and the estimate

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega')} \leq C\Big(\|u\|_{L^p(\Omega')} + \{u\}_{\mathcal{L}^{p,\lambda}(\Omega')}\Big)$$

holds for a constant $C = C(p, \lambda, \Omega', \Omega'') > 0$ that does not depend on u.

Exercise 11.2 Let $\Omega \subset \mathbb{R}^n$ be a bounded, regular domain and let $\Delta^2 \varphi = \Delta(\Delta\varphi)$ be the Bilaplacian. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary and

$$\Xi := \{u \in H^4(\Omega) \cap H^1_0(\Omega) \mid \Delta u \in H^1_0(\Omega)\}.$$

(i) Prove that the Bilaplacian

$$\Delta^2 \colon \Xi \to L^2(\Omega), \quad u \mapsto \Delta(\Delta u),$$

is bijective from Ξ onto $L^2(\Omega)$.

(ii) Given $f \in L^2(\Omega)$, let $u \in \Xi$ satisfy $\Delta^2 u = f$. Prove that for every $\varphi \in \Xi$ there holds

$$\int_\Omega u \Delta^2 \varphi \, dx = \int_\Omega f \varphi \, dx. \tag{$\dagger$}$$

(iii) Assume that $u, f \in L^2(\Omega)$ satisfy (**??**). Prove that $u \in \Xi$.

Exercise 11.3 We revisit Exercise 10.5 with the notions ellpitic regularity theory we have acquired.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary.

(i) Prove that

$$\langle u, v \rangle := \int_\Omega \Delta u \Delta v \, dx$$

defines a scalar product on $H^2(\Omega) \cap H^1_0(\Omega)$ which is equivalent to the standard scalar product $(\cdot, \cdot)_{H^2(\Omega)}$.

(ii) Show that $\big(H^2(\Omega) \cap H^1_0(\Omega), \langle \cdot, \cdot \rangle\big)$ is a Hilbert space.

(iii) Prove that given $f \in L^2(\Omega)$ there is a unique $u \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfying

$$\forall v \in H^2(\Omega) \cap H^1_0(\Omega) : \quad \int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx.$$

Show that in fact $u \in \Xi := \{u \in H^4(\Omega) \cap H^1_0(\Omega) \mid \Delta u \in H^1_0(\Omega)\}$ and $\Delta^2 u = f$.

Exercise 11.4 Let $\Omega \subseteq \mathbb{R}^n$ be a regular domain. Consider a function $u \in H^1_0(\Omega)$ so that its weak Laplacian Δu is in $L^2(\Omega)$, namely, there exists $f \in L^2(\Omega)$ so that for every $\varphi \in C^\infty_c(\Omega)$ there holds

$$\int_\Omega u \Delta \varphi \, dx = \int_\Omega f \varphi \, dx,$$

for which we then set $f = \Delta u$. By definition, there exist a sequence $u \in C^\infty_c(\Omega)$ so that

$$\lim_{k \to \infty} u_k = u \quad \text{in } H^1(\Omega)$$

We ask whether the sequence can be chosen so that *additionally* it satisfies

$$\lim_{k \to \infty} \Delta u_k = \Delta u \quad \text{in } L^2(\Omega).$$

(i) Prove that the answer is positive when $\Omega = \mathbb{R}^n$.

(ii) Prove that the answer is, in general, negative for $u \in H^1_0(\Omega)$ when Ω is bounded.

(iii) Can you characterize the subset of functions $u \in H^1_0(\Omega)$ for which such approximating sequence exists?

Exercise 11.5 Let $\Omega \subset \mathbb{R}^n$ be open. Let $a^{ij} \colon \Omega \to \mathbb{R}$ be measurable functions for every $i, j \in \{1, \ldots, n\}$. A differential operator L in non-divergence form

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x)\partial_{ij}^2 u,$$

is called *uniformly elliptic in* Ω, if there exists $\lambda > 0$ such that for almost every $x \in \Omega$ and every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2. \tag{1}$$

For $a^{ij} \in C^2(\overline{\Omega})$ and $c \in C^0(\overline{\Omega})$, we say that $u \in H_0^1(\Omega)$ is a *weak solution* of

$$-Lu + cu = f \quad \text{in } \Omega, \tag{2}$$

if for every $\varphi \in H_0^1(\Omega)$ there holds

$$\sum_{i,j=1}^{n} \int_\Omega a^{ij}\partial_j u\, \partial_i \varphi + \partial_i a^{ij}\, \partial_j u\, \varphi\, dx + \int_\Omega cu\varphi\, dx = \int_\Omega f\varphi\, dx. \tag{3}$$

(i) Prove that a classical solution $u \in C^2(\Omega) \cap H_0^1(\Omega)$ of $-Lu + cu = f$ is also a weak solution.

(ii) Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $f \in L^2(\Omega)$. Let $a^{ij} \in C^2(\overline{\Omega})$ satisfy (1). Find a condition on $c \in C^0(\overline{\Omega})$ so that (2) admits a unique weak solution $u \in H_0^1(\Omega)$.

Exercise 11.6 Given $u \in H^2(\mathbb{R}^n_+) \cap H_0^1(\mathbb{R}^n_+)$ prove that

$$\frac{\partial u}{\partial x_i} \in H_0^1(\mathbb{R}^n_+)$$

for every $i \in \{1, \ldots, n-1\}$.

Hints to Exercises.

11.1 For (i), use the p–triangle inequality $|a+b|^p \leq C_p(|a|^p + |b|^p)$

For (ii) recall also Exercise 9.1.

11.2 For (i), construct the sequence explicitly;

For (ii), use the elliptic estimates.

11.5 Seek to apply the Lax-Milgram theorem.

11.6 Argue with the difference quotients.