Multiple Choice Questions

The following exercise sheet list a number of multiple choice questions that were given over the past years; they may be helpful in preparing for the exam or to test your knowledge of the subject. Some are similar to exercises of the previous sheets.

For each question, only one is the correct answer; justify your choice.

13.1. For what values of p is $u\colon]{-1},1[\to \mathbb{R}$ given by $u(x) = |x|$ in $W^{1,p}(]{-1},1[)$?

(a) only for $p = 1$.

(b) only for $p = 1$ and $p = 2$.

(c) for all $p \in [1, \infty[$ but not for $p = \infty$.

(d) for all $p \in [1, \infty]$.

(e) None of the above.

13.2. For what values of p is $u\colon \mathbb{R} \to \mathbb{R}$ given by $u(x) = |x|$ in $W^{1,p}(\mathbb{R})$?

(a) only for $p = 1$.

(b) only for $p = 1$ and $p = 2$.

(c) for all $p \in [1, \infty[$ but not for $p = \infty$.

(d) for all $p \in [1, \infty]$.

(e) None of the above.

13.3. Let $n \in \mathbb{N}$ and $B_{\frac{1}{2}} = \{x \in \mathbb{R}^n \mid |x| < \frac{1}{2}\}$. Given $\alpha \in \mathbb{R}$, let $u_\alpha(x) = \Big|\log|x|\Big|^\alpha$. What is the set A_n of all $\alpha \in \mathbb{R}$ depending on n such that $u_\alpha \in W^{1,2}(B_{\frac{1}{2}})$?

(a) $A_1 = \{0\}, \quad A_2 =]-\infty, \frac{1}{2}[, \quad A_n = \mathbb{R}$ if $n \geq 3$.

(b) $A_1 = \mathbb{R}, \quad A_2 =]-\infty, \frac{1}{2}[, \quad A_n = \{0\}$ if $n \geq 3$.

(c) $A_n =]-\infty, \frac{n}{2}[$ for any $n \in \mathbb{N}$.

(d) $A_n =]-\infty, 0]$ for any $n \in \mathbb{N}$.

(e) $A_n = \{0\}$ for any $n \in \mathbb{N}$.

13.4. Let $n \in \mathbb{N}$ and $B_{\frac{1}{2}} = \{x \in \mathbb{R}^n \mid |x| < \frac{1}{2}\}$. Given $\alpha \in \mathbb{R}$, let $u_\alpha(x) = \Big|\log|x|\Big|^\alpha$. What is the set B_n of all $\alpha \in \mathbb{R}$ depending on n such that $u_\alpha \in W^{1,\infty}(B_{\frac{1}{2}})$?

(a) $B_1 = \{0\}, \quad B_2 =]-\infty, \frac{1}{2}[, \quad B_n = \mathbb{R}$ if $n \geq 3$.

(b) $B_1 = \mathbb{R}, \quad B_2 =]-\infty, \frac{1}{2}[, \quad B_n = \{0\}$ if $n \geq 3$.

(c) $B_n =]-\infty, \frac{n}{2}[$ for any $n \in \mathbb{N}$.

(d) $B_n =]-\infty, 0]$ for any $n \in \mathbb{N}$.

(e) $B_n = \{0\}$ for any $n \in \mathbb{N}$.

13.5. Let $f(x_1, x_2) = x_1 \sin(\frac{1}{x_1}) + x_2 \sin(\frac{1}{x_2})$. Which of the following is true?

(a) $\dfrac{\partial f}{\partial x_1} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ exists as weak derivative.

(b) $\dfrac{\partial f}{\partial x_2} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ exists as weak derivative.

(c) $\dfrac{\partial^2 f}{\partial x_1 \partial x_2} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ exists as weak derivative.

(d) All of the above.

(e) None of the above.

13.6. Let $\mathbb{R}_+ = \,]0,\infty[\, \subset \mathbb{R}$. Any $u \in W^{1,2}(\mathbb{R}_+)$ has a bounded representative.

(a) True.

(b) False.

13.7. The weak derivative of any $u \in W^{1,2}(\mathbb{R}_+)$ has a bounded representative.

(a) True.

(b) False.

13.8. Let $I := \,]a,b[$ for $-\infty < a < b < \infty$. Then the boundary-value problem

$$\begin{cases} -u'' + u' = f & \text{in } I, \\ \quad u'(a) = 0 = u'(b) \end{cases}$$

has *at least* one weak solution $u \in H^1(I)$ for every $f \in C^0(\overline{I})$.

(a) True.

(b) False.

13.9. Let $I := \,]a,b[$ for $-\infty < a < b < \infty$. Then the boundary-value problem

$$\begin{cases} -u'' + u' = f & \text{in } I, \\ \quad u'(a) = 0 = u'(b) \end{cases}$$

has *at most* one weak solution $u \in H^1(I)$ for every $f \in C^0(\overline{I})$.

(a) True.

(b) False.

13.10. Let $n \in \mathbb{N}$ and $B_{\frac{1}{2}} = \{x \in \mathbb{R}^n \mid |x| < \frac{1}{2}\}$. Given $\alpha \in \mathbb{R}$, let $u_\alpha(x) = \Big|\log|x|\Big|^\alpha$. If α is chosen such that $u_\alpha \in W^{1,2}(B_{\frac{1}{2}})$, then u_α has a representative in $C^0(\overline{B_{\frac{1}{2}}})$.

(a) True.

(b) False.

13.11. Let $n = 1$ and $B_{\frac{1}{2}} = \{x \in \mathbb{R} \mid |x| < \frac{1}{2}\}$. Given $\alpha \in \mathbb{R}$, let $u_\alpha(x) = \Big|\log|x|\Big|^\alpha$. If α is chosen such that $u_\alpha \in W^{1,2}(B_{\frac{1}{2}})$, then u_α has a representative in $C^1(\overline{B_{\frac{1}{2}}})$.

(a) True.

(b) False.

13.12. Let $I =]-1, 1[$ and let $u, v \colon I \to \mathbb{R}$ be given by $u(x) = |x|$ and $v(x) = (1-x^2)^{\frac{3}{4}}$. Then $uv \in W^{1,3}(I)$.

(a) True.

(b) False.

13.13. The Cantor function on $[0, 1]$ is absolutely continuous.

(a) True.

(b) False.

13.14. $\exists C > 0 \quad \forall u \in H^1(]0,1[) : \quad \displaystyle\int_0^1 |u|^2\,dx \leq C \int_0^1 |u'|^2\,dx.$

(a) True.

(b) False.

13.15. Let $0 < a < 1 < b < \infty$ such that $\int_a^b (\log x)\,dx = 0$. Then $\int_a^b |\log x|^2\,dx \leq \frac{(b-a)^3}{ab}$.

(a) True.

(b) False.

13.16. Let $\Omega \subset \mathbb{R}^3$ be open and bounded of class C^1. Into which space does $H^1(\Omega)$ *not* embed continuously?

(a) $C^0(\overline{\Omega})$

(b) $L^4(\Omega)$

(c) $L^6(\Omega)$

(d) $W^{1,1}(\Omega)$

(e) None of the above.

13.17. Let $n \geq 2$ and $p = 2n$. Into which space does $W^{1,p}(\mathbb{R}^n)$ *not* embed continuously?

(a) $L^\infty(\mathbb{R}^n)$

(b) $\mathcal{L}^{p,p}(\mathbb{R}^n)$

(c) $L^n(\mathbb{R}^n)$

(d) $C^{0,\frac{1}{2}}(\mathbb{R}^n)$

(e) None of the above.

13.18. Let $n \geq 2$ and $1 < p < n$. Let $\Omega \subset \mathbb{R}^n$ be bounded of class C^1. Let $u \in W^{1,p}(\Omega)$. Which statement is false?

(a) $u|_{\partial\Omega} \in L^p(\partial\Omega)$ is well-defined.

(b) The embedding $W^{1,p}(\Omega) \hookrightarrow L^{\frac{n}{n-p}}(\Omega)$ is compact.

(c) There exists $C < \infty$ independently of u such that $\|u\|_{W^{1,p}(\Omega)} \leq C\|u\|_{L^{\frac{np}{n-p}}(\Omega)}$.

(d) There exists more than one $v \in W^{1,p}(\mathbb{R}^n)$ with $v|_\Omega = u$.

(e) None of the above.

13.19. Let $n \geq 3$ and let $B_1 \subset \mathbb{R}^n$ be the unit ball. For which $q \geq 1$ is the following inequality true?

$$\exists C < \infty \quad \forall u \in C_c^\infty(B_1): \qquad \int_{B_1} |u|^3\,dx \leq C\Big(\int_{B_1} |\nabla u|^2\,dx\Big)\Big(\int_{B_1} |u|^q\,dx\Big)^{\frac{2}{n}}$$

(a) any $q \geq 3$

(b) only for $q = 3$

(c) only for $q = \frac{n}{3}$

(d) only for $q = \frac{n}{2}$

(e) None of the above.

13.20. Let $n \in \mathbb{N}$ and let $B_R \subset \mathbb{R}^n$ be the ball of radius $R > 0$ around the origin. Let $1 \leq p < n$ and let $1 \leq q \leq \frac{np}{n-p}$. For which $\beta \in \mathbb{R}$ is the following statement true?

$$\exists C < \infty \quad \forall R > 0 \quad \forall u \in W_0^{1,p}(B_R): \qquad \|u\|_{L^q(B_R)} \leq CR^\beta \|\nabla u\|_{L^p(B_R)}$$

(a) $\beta = 0$

(b) $\beta = 1$

(c) $\beta = \frac{n-p}{q}$

(d) $\beta = \frac{n}{q} - \frac{n}{p} + 1$

(e) None of the above.

13.21. Let $u \in W^{1,1}(\mathbb{R}^n)$. Let $\Omega \subset \mathbb{R}^n$ be any open domain. Then $u|_\Omega \in W^{1,1}(\Omega)$.

(a) True.

(b) False.

13.22. Let $n \in \mathbb{N}$ and $1 \leq p < \infty$. The spaces $W^{1,p}(\mathbb{R}^n)$ and $W_0^{1,p}(\mathbb{R}^n)$ are the same.

(a) True.

(b) False.

13.23. Let $n \geq 2$ and $1 \leq p < n$. The spaces $W^{1,p}(\mathbb{R}^n \setminus \overline{B_1(0)})$ and $W_0^{1,p}(\mathbb{R}^n \setminus \overline{B_1(0)})$ are the same.

(a) True.

(b) False.

13.24. Let $n \geq 2$ and $1 \leq p < n$. The spaces $W^{1,p}(\mathbb{R}^n \setminus \{0\})$ and $W_0^{1,p}(\mathbb{R}^n \setminus \{0\})$ are the same.

(a) True.

(b) False.

13.25. Let $n = 1$ and $p = 1$. The spaces $W^{1,1}(\mathbb{R} \setminus \{0\})$ and $W_0^{1,1}(\mathbb{R} \setminus \{0\})$ are the same.

(a) True.

(b) False.

13.26. Let $\Omega \subset \mathbb{R}^n$ be bounded of class C^1. Then, $C^\infty(\overline{\Omega})$ is dense in $C^{0,\frac{1}{2}}(\overline{\Omega})$.

(a) True.

(b) False.

13.27. Let $B_1 \subset \mathbb{R}^2$ be the unit ball and let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $W^{1,4}(B_1)$. Then, there exists $v \in C^{0,\frac{1}{4}}(\overline{B_1})$ and $\Lambda \subset \mathbb{N}$ such that $\|v - u_k\|_{C^{0,\frac{1}{4}}(\overline{B_1})} \to 0$ as $\Lambda \ni k \to \infty$.

(a) True.

(b) False.

13.28. There exists functions in $W^{1,3}(\mathbb{R}^2)$ such that all their representatives are nowhere differentiable.

(a) True.

(b) False.

13.29. Every compactly supported function in $W^{1,4}(\mathbb{R}^3)$ is in $L^\infty(\mathbb{R}^3)$.

(a) True.

(b) False.

13.30. Let $B_1 \subset \mathbb{R}^n$ be the unit ball. There exists a sequence $(u_k)_{k\in\mathbb{N}}$ in $C_c^\infty(B_1)$ satisfying $u_k(x) = 1$ for every $x \in B_1$ with $|x| = \frac{1}{2}$ and $\|u_k\|_{W^{1,1}(B_1)} \to 0$ as $k \to \infty$.

(a) True.

(b) False.

13.31. Let $B_1 \subset \mathbb{R}^4$ be the unit ball and let $f \in H^k(B_1)$. Let $u \in H_0^1(B_1)$ be a weak solution of $-\Delta u = f$ in B_1. What is the minimal value of $k \in \mathbb{N}$ that guarantees u to be a classical solution in $C^2(\overline{B_1})$?

(a) $k = 1$

(b) $k = 2$

(c) $k = 3$

(d) $k = 4$

(e) None of the above.

13.32. Let $B_1 \subset \mathbb{R}^n$ be the unit ball and let $f \in H^k(B_1)$ for some $k \in \mathbb{N}$. Let $u \in H_0^1(B_1)$ be a weak solution of $-\Delta u = f$ in B_1. Then u is always bounded in B_1 provided

(a) $n > \frac{k}{2} + 1$

(b) $n > 2k$

(c) $n < 4k + 2$

(d) $n < 2k + 4$

(e) None of the above.

13.33. Given $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a weak solution of $-\Delta u = f$ in $\mathbb{R}^n$. For what $\alpha, \beta \in \mathbb{R}$ is the following statement true? (with C independent of u)

$$\exists C < \infty \quad \forall R > 0 : \quad \int_{B_R} |\nabla u|^2 \, dx \leq C\Big(R^\alpha \int_{B_{2R}} |u|^2 \, dx + R^\beta \int_{B_{2R}} |f|^2 \, dx \Big).$$

(a) $\alpha = 2$ and $\beta = 2$

(b) $\alpha = -2$ and $\beta = 2$

(c) $\alpha = 2$ and $\beta = -2$

(d) $\alpha = -2$ and $\beta = -2$

(e) None of the above.

13.34. Let $\Omega \subset\subset \Omega' \subset\subset \mathbb{R}^n$ be smooth domains. Let $\lambda(\Omega)$ respectively $\lambda(\Omega')$ be the corresponding first (smallest) Dirichlet eigenvalue for $-\Delta$ in Ω respectively Ω'. Then

(a) $\lambda(\Omega) < \lambda(\Omega')$

(b) $\lambda(\Omega) \leq \lambda(\Omega')$ and equality may occur

(c) $\lambda(\Omega) > \lambda(\Omega')$

(d) $\lambda(\Omega) \geq \lambda(\Omega')$ and equality may occur

(e) None of the above.

13.35. In which domain Ω is the following differential operator uniformly elliptic?

$$Lu = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big), \qquad \big(a_{ij}(x_1,x_2)\big) = \begin{pmatrix} x_1^2 + x_2^2 & x_1 + x_2 \\ x_1 + x_2 & 1 \end{pmatrix}$$

(Here, $B_1(p) = \{x \in \mathbb{R}^2 \mid |x-p| < 1\}$ denotes the unit ball around $p \in \mathbb{R}^2$.)

(a) $\Omega = B_1((0,1))$

(b) $\Omega = B_1((1,0))$

(c) $\Omega = B_1((0,-1))$

(d) $\Omega = B_1((-1,0))$

(e) None of the above.

13.36. Let $B_1 \subset \mathbb{R}^3$ be the unit ball. Then, there exists a unique $u \in C^3(\overline{B_1}) \cap H_0^1(B_1)$ satisfying $\Delta u = -1$ in B_1.

(a) True.

(b) False.

13.37. Let $\Omega \subset \mathbb{R}^n$ be open. Let L be uniformly elliptic and bounded in divergence form with smooth coefficients. If $u \in H^1_{\mathrm{loc}}(\Omega)$ is a weak solution of $-Lu = f$ and $f \in L^2_{\mathrm{loc}}(\Omega)$, then $u \in H^2_{\mathrm{loc}}(\Omega)$.

(a) True.

(b) False.

13.38. Let $\Omega \subset \mathbb{R}^n$ be open and bounded of class C^∞. There exists $m \in \mathbb{N}$ such that the embedding $W^{k,p}(\Omega) \hookrightarrow C^m(\Omega)$ does *not* hold for any $k, p \in \mathbb{N}$.

(a) True.

(b) False.

13.39. Let $\Omega \subset \mathbb{R}^n$ be open and bounded of class C^∞. For sufficiently large $k, p \in \mathbb{N}$ the embedding $W^{k,p}(\Omega) \hookrightarrow C^\ell$ holds for any $\ell \in \mathbb{N}$.

(a) True.

(b) False.

13.40. The three smallest Dirichlet eigenvalues $0 < \lambda_1, \lambda_2, \lambda_3$ of the Laplacian $-\Delta$ on the rectangle $D =]0, \pi[\times]0, 3\pi[\subset \mathbb{R}^2$ are

$$\lambda_1 = 1, \qquad \lambda_2 = \tfrac{13}{9}, \qquad \lambda_3 = 2.$$

(a) True.

(b) False.

13.41. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $f \in L^2(\Omega)$. Let $u \in H^1(\Omega)$ be a weak solution of $-\Delta u = f$. Then, for any $\Omega' \subset\subset \Omega$, there exists a constant $C < \infty$ which depends only on the pair Ω, Ω' (but not on u nor k) such that

$$\forall k \in \mathbb{R}: \quad \int_{\Omega'} |\nabla u|^2 \, dx \leq C\Big(\int_\Omega |u-k|^2 \, dx + \int_\Omega |f|^2 \, dx\Big).$$

(a) True.

(b) False.

13.42. Let $\Omega \subset \mathbb{R}^2$ be open and bounded. Provided we have a solution $u \in C^2(\overline{\Omega})$ of the *minimal surface equation*

$$(1+u_1^2)\, u_{22} - 2u_1 u_2\, u_{12} + (1+u_2^2)\, u_{11} = 0 \quad \text{in } \Omega,$$

where the subscripts denote partial derivatives, then this equation can be rewritten in divergence form.

(a) True.

(b) False.

13.43. Let $B_1 \subset \mathbb{R}^2$ be the unit disc. Then, the problem

$$\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) = 1, \qquad\qquad \big(a_{ij}(x_1,x_2)\big) = \begin{pmatrix} 2 & \frac{x_1 x_2}{|x_1 x_2|} \\ \frac{x_1 x_2}{|x_1 x_2|} & 2 \end{pmatrix}$$

has a weak solution $u \in H_0^1(B_1)$.

(a) True.

(b) False.

13.44. Assume $u \in H^1(\Omega)$ is harmonic, namely solves $-\Delta u = 0$ (weakly and thus classically) on a bounded smooth domain Ω and $g := u|_{\partial\Omega} \in C^0(\partial\Omega)$. Then:

1. $u \in L^\infty(\Omega)$.

   (a) True.

   (b) False.

2. $u \in C^0(\overline{\Omega})$.

   (c) True.

   (d) False.

13.45. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $V \in C^\infty(\overline{\Omega})$. Recall that the operator $A = -\Delta + V \colon D_A \subset L^2(\Omega) \to L^2(\Omega)$, where $D_A = \{u \in C^2(\overline{\Omega}) \mid u|_{\partial\Omega} = 0\}$ is closable. The domain of its closure $\overline{A}$ is $D_{\overline{A}} = H^2(\Omega) \cap H_0^1(\Omega)$.

(a) True.

(b) False.

13.46. Let $Q =]0,1[^n \subset \mathbb{R}^n$ and let $\Gamma = \{0,1\}^n \subset \mathbb{R}^n$ be the set of vertices of Q. Let $u \in H_0^1(Q)$ be a weak solution of $-\Delta u = \lambda u$ in Q for some $\lambda \in \mathbb{R}$. Then,

When $n = 2$, $u \in C^\infty(\overline{Q})$.

(a) True.

(b) False.

For $n \geq 2$,

(c) $u \in C^\infty(Q)$ but $u \notin C^\infty(\overline{Q})$

(d) $u \in C^\infty(\overline{Q} \setminus \Gamma)$ but $u \notin C^\infty(\overline{Q})$

(e) $u \in C^\infty(\overline{Q})$

(f) The answer depends on the dimension $n \in \mathbb{N}$.

(g) None of the above.

13.47. (Requires the use of *Hopf's Lemma*) Let $\Omega \subset \mathbb{R}^n$ be a smooth and connected open domain. Suppose $u \in C^2(\overline{\Omega})$ satisfies $u \geq 0$, $u \not\equiv 0$, $u|_{\partial\Omega} = 0$ and $\Delta u + \lambda u = 0$ for some $\lambda \in \mathbb{R}$. Then,

(a) λ is the largest negative Dirichlet eigenvalue of the operator $-\Delta$.

(b) $\lambda = 0$.

(c) λ is the smallest positive Dirichlet eigenvalue of the operator $-\Delta$.

(d) λ is a positive eigenvalue of $-\Delta$, but not necessarily the smallest one.

(e) None of the above.

13.48. For which $n \in \mathbb{N}$ is the following statement true? Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain of class C^1. Then,

$$\forall \varepsilon > 0 \quad \exists C < \infty \quad \forall u \in H^1(\Omega): \quad \|u\|_{L^4(\Omega)} \leq \varepsilon \|\nabla u\|_{L^2(\Omega)} + C\|u\|_{L^2(\Omega)}$$

(a) for all $n \in \mathbb{N}$.

(b) only for $n \in \{1, 2, 3, 4\}$.

(c) only for $n \in \{1, 2, 3\}$.

(d) only for $n \in \{2, 3\}$.

(e) None of the above.

13.49. Assume $-\Delta u = f$ in the open unit ball $B_1 \subset \mathbb{R}^n$ for some function $f \in C^0(\overline{B_1})$ satisfying $|f(x)| \leq 1$ for all $x \in \overline{B_1}$ and $|f| \not\equiv 1$ and some $u \in C^2(\overline{B_1})$ vanishing along the boundary ∂B_1. Then, for every $x \in B_1$

(a) $|u(x)| < \frac{1}{2n}|x|^2$.

(b) $|u(x)| \leq \frac{1}{2n}|x|^2$ and equality may occur at some point in B_1.

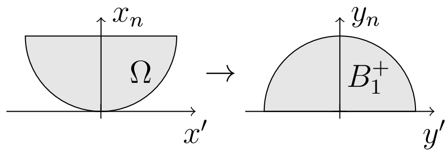
(c) $|u(x)| < \frac{1}{2n}\Big(1 - |x|^2\Big)$.

(d) $|u(x)| \leq \frac{1}{2n}\Big(1 - |x|^2\Big)$ and equality may occur at some point in B_1.

(e) None of the above.

13.50. Given $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$ we define $x' = (x^1, x^2, \ldots, x^{n-1}) \in \mathbb{R}^{n-1}$. Consider the domain $\Omega = \{x = (x', x^n) \in \mathbb{R}^n \mid |(x', x^n - 1)| < 1,\ x^n < 1\}$ and let

$$\begin{aligned}\Phi\colon \Omega &\to B_1^+\\ (x', x^n) &\mapsto \Big(x', x^n - 1 + \sqrt{1 - |x'|^2}\Big)\end{aligned}$$



be a diffeomorphism flattening the lower part $\Gamma = \partial\Omega \cap \partial B_1(0,1)$ of the boundary of Ω. Let $u \in H^1(\Omega)$ be a weak solution with vanishing trace on Γ of the equation

$$-\Delta u = \operatorname{div} f$$

where $f = (f^1, \ldots, f^n)^\intercal \in C^{1,\alpha}(\Omega; \mathbb{R}^n)$. Then, the equation solved by $v := u \circ \Phi^{-1}$ is

$$-\operatorname{div}(a \cdot \nabla v) = \operatorname{div}\Big(b \cdot (f \circ \Phi^{-1})\Big)$$

with $(n \times n)$-matrices $a = a(y)$ and $b = b(y)$ given by

(a) $a = \mathrm{Id}_{\mathbb{R}^n}, \qquad b = \mathrm{Id}_{\mathbb{R}^n}$

(b) $a = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & \frac{(y')^\intercal}{\sqrt{1-|y'|^2}} \\ \frac{y'}{\sqrt{1-|y'|^2}} & \frac{1+|y'|^2}{1-|y'|^2} \end{pmatrix}, \qquad b = \mathrm{Id}_{\mathbb{R}^n}$

(c) $a = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & \frac{(y')^\intercal}{\sqrt{1-|y'|^2}} \\ \frac{y'}{\sqrt{1-|y'|^2}} & \frac{1}{1-|y'|^2} \end{pmatrix}, \qquad b = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & \frac{(y')^\intercal}{\sqrt{1-|y'|^2}} \\ 0 & 1 \end{pmatrix}$

(d) $a = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & \frac{-(y')^\intercal}{\sqrt{1-|y'|^2}} \\ \frac{-y'}{\sqrt{1-|y'|^2}} & \frac{1}{1-|y'|^2} \end{pmatrix}, \qquad b = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & 0 \\ \frac{-y'}{\sqrt{1-|y'|^2}} & 1 \end{pmatrix}$

(e) None of the above.

13.51. Let $2 \leq n \in \mathbb{N}$, let $\Omega := B_{\frac{1}{2}}(0) \subset \mathbb{R}^n$ and let $g\colon \Omega \to \mathbb{R}$ be given by $g(x) = \log\log(\frac{1}{|x|})$. Then, g is in the Campanato space $\mathcal{L}^{2,n}(\Omega)$.

(a) True.

(b) False.

13.52. Let $2 \leq n \in \mathbb{N}$, let $\Omega := B_{\frac{1}{2}}(0) \subset \mathbb{R}^n$ and let $g\colon \Omega \to \mathbb{R}$ be given by $g(x) = \log\log(\frac{1}{|x|})$. Then, g is in the Morrey space $L^{2,n}(\Omega)$.

(a) True.

(b) False.

13.53. Let $2 \leq n \in \mathbb{N}$ and $\Omega := B_{\frac{1}{2}} \subset \mathbb{R}^n$. Then, $L^{2,n}(\Omega) \subsetneq \mathcal{L}^{2,n}(\Omega)$.

(a) True.

(b) False.

13.54. Let $\Omega \subset \mathbb{R}^n$ be any open, bounded domain. Then,

$$\exists C < \infty \quad \forall u \in C^{1,\frac{1}{2}}(\overline{\Omega}): \quad \|u\|_{C^1(\overline{\Omega})} \leq \tfrac{1}{9}\|u\|_{C^{1,\frac{1}{2}}(\overline{\Omega})} + C\|u\|_{H^1(\Omega)}.$$

(a) True.

(b) False.

13.55. Let $\Omega \subset \mathbb{R}^n$ be any bounded domain. Given $0 < \alpha < 1$ let $f \in C^{0,\alpha}(\overline{\Omega})$. Then there exists $g \in C_c^{0,\alpha}(\mathbb{R}^n)$ such that $g|_\Omega = f$.

(a) True.

(b) False.

13.56. Let $\Omega \subset\subset \mathbb{R}^n$ be any open, bounded, smooth domain and let $u \in H_0^1(\Omega)$ be the unique weak solution of $\Delta u = (1 + |x|^2)$ in Ω. Then, $u(x) \neq 0$ for every $x \in \Omega$.

(a) True.

(b) False.

13.57. Let $\Omega \subset\subset \mathbb{R}^n$ be any open, bounded, smooth domain and let $u \in H_0^1(\Omega)$ be the unique weak solution of $\Delta u = (1 + |x|^2)$ in Ω. Then, $u \geq 0$ in $\overline{\Omega}$.

(a) True.

(b) False.

13.58. Let $B_1 \subset \mathbb{R}^2$ be the unit disc. Let $u \in C^{1,\alpha}(B_1)$ satisfy

$$\forall \varphi \in C_c^\infty(B_1): \quad \int_{B_1} \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1+|\nabla u|^2}}\,dx = 0.$$

Then, $u \in C^2(B_1)$.

(a) True.

(b) False.

13.59. (Requires the use of *Hopf's Lemma*) If a non-negative function $u \in C^2(\overline{B_1})$ solves $\Delta u = cu$ for some $c \in C^0(\overline{B_1})$ and satisfies $u > 0$ on ∂B_1, then $u > 0$ in B_1.

(a) True.

(b) False.

13.60. (Requires the use of *interior Schauder estimates*) If $u \in H^1(B_1) \cap C^{0,\alpha}_{\mathrm{loc}}(B_1)$ weakly solves $-\Delta u + cu = 0$ in the unit ball $B_1 \subset \mathbb{R}^n$ for some function $c \in C^{6,\alpha}_{\mathrm{loc}}(B_1)$ then $u \in C^{8,\alpha}_{\mathrm{loc}}(B_1)$.

(a) True.

(b) False.