Exercise 1.1 Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$ and let $\Omega :=]a, b[\times]c, d[\subset \mathbb{R}^2$.

(i) Find all the harmonic functions on Ω that are of the form $u(x, y) = v(x)w(y)$.

You may appeal to the *unique continuation principle* stating that if $u, \tilde{u} \in C^2(\Omega)$ satisfy $\Delta u = 0 = \Delta \tilde{u}$ and $u|_Q = \tilde{u}|_Q$ for some open set $Q \subset \Omega$, then $u = \tilde{u}$ in Ω.

(ii) Prove or disprove: For every $u_0 \in C^2(\partial\Omega)$ there is a solution of the form $u(x, y) = v(x)w(y)$ to the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

Solution. (i) If $u \in C^2(\Omega)$ is of the form $u(x, y) = v(x)w(y)$, then

$$(\Delta u)(x, y) = v''(x)\, w(y) + v(x)\, w''(y).$$

Suppose that $\Delta u = 0$. At every $(x, y) \in \Omega$ where $v(x)w(y) \neq 0$ we obtain

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)}. \tag{‡}$$

Since the left hand side depends only on x and the right hand side only on y, the equation requires both sides to be constant. More precisely,

$$\frac{v''(x)}{v(x)} = \kappa = -\frac{w''(y)}{w(y)}$$

at every $(x, y) \in \Omega$ where $v(x)w(y) \neq 0$. The resulting equations

$$v''(x) = \kappa v(x), \qquad\qquad w''(y) = -\kappa w(y)$$

can be solved separately by distinguishing three cases.

Case 1. $\kappa = \lambda^2$ for some $\lambda > 0$. Then, with constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$

$$v(x) = C_1 \mathrm{e}^{\lambda x} + C_2 \mathrm{e}^{-\lambda x}, \qquad\qquad w(y) = C_3 \sin(\lambda y) + C_4 \cos(\lambda y).$$

Case 2. $\kappa = 0$. Then, with constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$

$$v(x) = C_1 x + C_2, \qquad\qquad w(y) = C_3 y + C_4.$$

Case 3. $\kappa = -\lambda^2$ for some $\lambda > 0$. Then, with constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$

$$v(x) = C_1 \sin(\lambda x) + C_2 \cos(\lambda x), \qquad w(y) = C_3 \mathrm{e}^{\lambda y} + C_4 \mathrm{e}^{-\lambda y}.$$

In each of the cases, one verifies directly that $u(x, y) = v(x)w(y)$ is harmonic.

Are these all harmonic functions of this form? Let $u(x, y) = v(x)w(y)$ in $C^2(\Omega)$ satisfy $\Delta u = 0$ in Ω. If u is not identically zero, there are open set $I \subset]a, b[$ and $J \subset]c, d[$ such that $v(x) \neq 0 \ \forall x \in I$ and $w(y) \neq 0 \ \forall y \in J$. Hence equation (‡) is satisfied in $I \times J$ and $u|_{I\times J}$ agrees with the restriction of one of the solutions $\tilde{u}$ found in cases 1–3. Since $I \times J$ is open, the unique continuation principle yields $u = \tilde{u}$ in Ω.

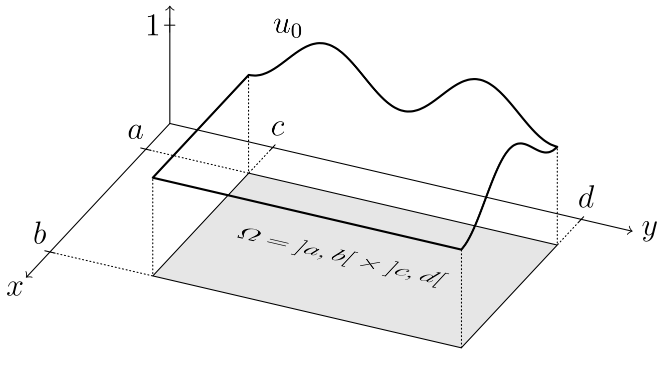
(ii) Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$ and let $\Omega :=]a, b[\times]c, d[\subset \mathbb{R}^2$. Let $u_0 \in C^2(\partial\Omega)$ be non-constant satisfying

$$\forall x \in [a, b] \quad u_0(x, c) = 1, \qquad\qquad \forall y \in [c, d] \quad u_0(b, y) = 1.$$

Then, any function $u(x, y) = v(x)w(y)$ in Ω with $u|_{\partial\Omega} = u_0$ must satisfy

$$\begin{aligned} \forall x \in [a, b] \quad & 1 = u_0(x, c) = u(x, c) = v(x)w(c) \quad && \Rightarrow \ v(x) = \tfrac{1}{w(c)}, \\ \forall y \in [c, d] \quad & 1 = u_0(b, y) = u(b, y) = v(b)w(y) \quad && \Rightarrow \ w(y) = \tfrac{1}{v(b)}. \end{aligned}$$

In particular, both v and w must be constant. This however is in contradiction to u_0 being non-constant.



$\square$

Exercise 1.2 We have seen that every harmonic function satisfies the mean value property. Prove that the converse is true: let $\Omega \subset \mathbb{R}^n$ be open and let $u \in C^2(\Omega)$ satisfy the mean-value property, i. e. for any $y \in \Omega$ and any $r > 0$ such that $B_r(y) \subset \Omega$,

$$u(y) = \fint_{\partial B_r(y)} u \, d\sigma = \fint_{B_r(y)} u \, dx.$$

Prove that u is harmonic.

Solution. For $R > 0$ so that $B_R(y) \subset \Omega$, define $\phi\colon]0, R[\to \mathbb{R}$ by

$$\phi(r) = \fint_{\partial B_r(y)} u \, d\sigma = \fint_{\partial B_1(0)} u(y + rz) \, d\sigma(z)$$

and compute

$$\begin{aligned} \phi'(r) &= \fint_{\partial B_1(0)} \frac{d}{dr}\Big(u(y+rz)\Big) \, d\sigma(z) = \fint_{\partial B_1(0)} z \cdot \nabla u(y+rz) \, d\sigma(z) \\ &= \fint_{\partial B_r(y)} \frac{\xi - y}{r} \cdot \nabla u(\xi) \, d\sigma(\xi) = \frac{r}{n} \fint_{B_r(y)} \Delta u \, dx, \end{aligned} \tag{†}$$

where the divergence theorem applies because $\nu = \frac{\xi - y}{r}$ is the outward unit normal vector along $\partial B_r(y)$. If u satisfies the mean-value property, ϕ is constant. In particular,

$$0 = \phi'(r) = \frac{r}{n} \fint_{B_r(y)} \Delta u \, dx. \tag{$*$}$$

By assumption, Δu is continuous. If $\Delta u \neq 0$, there exist $y \in \Omega$ and $r > 0$ such that either $\Delta u < 0$ in $B_r(y)$ or $\Delta u > 0$ in $B_r(y)$ which contradicts $(*)$ in both cases. □

Exercise 1.3 (Liouville's Theorem)

(i) Suppose $u \in C^2(\mathbb{R}^n)$ is harmonic and $u \in L^1(\mathbb{R}^n)$. Prove that $u = 0$.

(ii) Suppose $u \in C^2(\mathbb{R}^n)$ is harmonic and bounded. Prove that u is constant.

Solution. (i) Let $u \in C^2(\mathbb{R}^n)$ be harmonic and $u \in L^1(\mathbb{R}^n)$. Let $B_r(y) \subset \mathbb{R}^n$ be the open ball of radius $r > 0$ around y. By the mean-value property implies

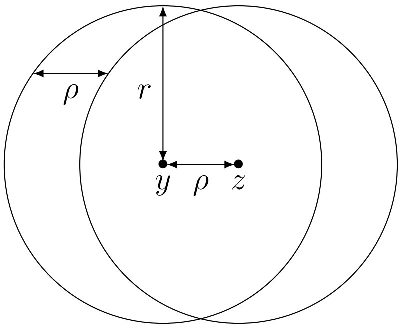
$$|u(y)| = \left| \fint_{B_r(y)} u \, dx \right| \leq \frac{1}{|B_r|} \int_{B_r(y)} |u| \, dx \leq \frac{1}{|B_r|} \|u\|_{L^1(\mathbb{R}^n)} \xrightarrow{r \to \infty} 0.$$

Since $y \in \mathbb{R}^n$ is arbitrary, we obtain $u \equiv 0$.

(ii) Let $u \in C^2(\mathbb{R}^n)$ be harmonic and $|u| \leq c_0$. Let $y, z \in \mathbb{R}^n$ be two arbitrary points and $\rho := |y - z|$. Then, for every $r > \rho$, the mean-value property implies

$$\begin{aligned} u(y) - u(z) &= \fint_{B_r(y)} u \, dx - \fint_{B_r(z)} u \, dx \\ &= \frac{1}{|B_r|} \int_{B_r(y) \setminus B_r(z)} u \, dx - \frac{1}{|B_r|} \int_{B_r(z) \setminus B_r(y)} u \, dx \\ &\leq \frac{2c_0}{|B_r|} |B_r(y) \setminus B_r(z)| \leq \frac{2c_0 \, \rho \, |B_r^{\mathbb{R}^{n-1}}|}{|B_r^{\mathbb{R}^n}|} \xrightarrow{r \to \infty} 0 \end{aligned}$$

i. e. $u(y) \leq u(z)$. Interchanging the roles of y and z gives $u(z) \leq u(y)$, i. e. $u(y) = u(z)$. Since $y, z \in \mathbb{R}^n$ are arbitrary, we conclude that u is constant.



□

Exercise 1.4 (Harnack's Inequality) Let $\Omega \subset \mathbb{R}^n$ be open. Let $Q \subset \Omega$ be any bounded and connected open subset such that $\overline{Q} \subset \Omega$. Prove that there exists a constant C depending only on Q such that for every *non-negative* harmonic function $u \in C^2(\Omega)$ there holds

$$\sup_Q u \leq C \inf_Q u.$$

Solution. Given the open set $\Omega \subset \mathbb{R}^n$ and the connected open subset $Q \subset \Omega$ such that $\overline{Q} \subset \Omega$, let $r = \frac{1}{4}\operatorname{dist}(Q, \partial\Omega) > 0$. Let $u \in C^2(\Omega)$ be harmonic. By the mean value property and since u is non-negative,

$$u(y) = \frac{1}{|B_{2r}|}\int_{B_{2r}(y)} u\,dx \geq \frac{1}{|B_{2r}|}\int_{B_r(z)} u\,dx = \frac{1}{2^n|B_r|}\int_{B_r(z)} u\,dx = \frac{1}{2^n}u(z)$$

for any $y, z \in Q$ with $|z - y| < r$. Since $\overline{Q}$ is connected and compact, there exist finitely many $x_1, \ldots, x_m \in Q$ such that $Q \subset \bigcup_{i=1}^m B_r(x_i)$ and such that $|x_i - x_{i+1}| < r$ for $i = 2, \ldots, m$. Consequently, for every $x, y \in Q$ there holds $u(x) \geq 2^{-n(m+1)}u(y)$, and thus

$$\sup_Q u \leq 2^{n(m+1)} \inf_Q u.$$

□