Exercise 3.1 Let $I :=]a, b[$ for $-\infty \le a < b \le \infty$. Let $u \in L^p(I)$ and let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in the Sobolev space $W^{1,p}(I)$ with $\|u - u_k\|_{L^p(I)} \to 0$ as $k \to \infty$.

(i) If $1 < p \le \infty$, prove $u \in W^{1,p}(I)$.

(ii) Is the assumption $p \neq 1$ in part (i) necessary?

Solution. (i) Let u_k' be the weak derivative of u_k. By assumption, the sequence $(u_k')_{k\in\mathbb{N}}$ is bounded in $L^p(I)$.

Case $1 < p < \infty$. In this case, the space $L^p(I)$ is reflexive and the Eberlein–Šmulyan Theorem applies: $(u_k')_{k\in\mathbb{N}}$ has a subsequence which converges weakly in $L^p(I)$. Let $g \in L^p(I)$ be the corresponding weak limit and $\Lambda \subset \mathbb{N}$ the subsequence's indices. Since for any $\varphi \in C_c^\infty(I)$, the maps $L^p(I) \to \mathbb{R}$ given by $f \mapsto \int_I f\varphi\,dx$ or by $f \mapsto -\int_I f\varphi'\,dx$ are elements of $(L^p(I))^*$ and since $\|u_k - u\|_{L^p} \to 0$ implies $u_k \xrightarrow{w} u$, we have by definition of weak convergence

$$-\int_I u\varphi'\,dx = \lim_{\Lambda\ni k\to\infty}\Big(-\int_I u_k\varphi'\,dx\Big) = \lim_{\Lambda\ni k\to\infty}\Big(\int_I u_k'\varphi\,dx\Big) = \int_I g\varphi\,dx$$

for any $\varphi \in C_c^\infty(I)$. Hence, $g \in L^p(I)$ is indeed the weak derivative of $u \in L^p(I)$ and $u \in W^{1,p}(I)$ follows.

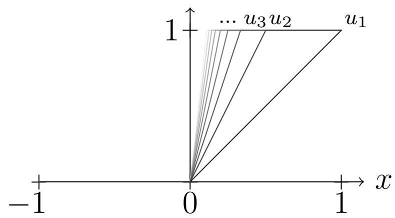
Case $p = \infty$. Since $L^1(I)$ is separable, the Banach–Alaoglu Theorem applies: $(u_k')_{k\in\mathbb{N}}$ being bounded in $L^\infty(I) \cong (L^1(I))^*$ has a subsequence (given by $\Lambda \subset \mathbb{N}$) which weak*-converges to some $g \in (L^1(I))^*$. For any $\varphi \in C_c^\infty(]0,1[) \subset L^1(]0,1[)$,

$$-\int_I u\varphi'\,dx = \lim_{\Lambda\ni k\to\infty}\Big(-\int_I u_k\varphi'\,dx\Big) = \lim_{\Lambda\ni k\to\infty}\Big(\int_I u_k'\varphi\,dx\Big) = \int_I g\varphi\,dx$$

follows as in part (i). Hence, $g \in (L^1(I))^* \cong L^\infty(I)$ is indeed the weak derivative of $u \in L^\infty(I)$ and $u \in W^{1,\infty}(I)$ follows.

(ii) The assumption $p \neq 1$ is necessary. Consider $I =]-1, 1[$ and $u = \chi_{]0,1[} \in L^1(I)$. For every $k \in \mathbb{N}$ let $u_k\colon I \to \mathbb{R}$ be given by

$$u_k(x) = \begin{cases} 0, & \text{for } -1 < x \le 0, \\ kx, & \text{for } 0 < x \le \frac{1}{k}, \\ 1, & \text{for } \frac{1}{k} < x \le 1. \end{cases}$$



Then, $u_k \in W^{1,1}(I)$ with $\|u_k\|_{L^1} = 1 - \frac{1}{2k}$ and $\|u_k'\|_{L^1} = \frac{1}{k}k = 1$. Moreover, there holds $\|u_k - u\|_{L^1} = \frac{1}{2k} \to 0$ as $k \to \infty$. However, $u \notin W^{1,1}(I)$, otherwise u would have a continuous representative.

□

Exercise 3.2 Consider the function $f(x) = \log|x|$. From one variable calculus we know that $f \in L^p((-1,1))$ for every $p \in [1,\infty)$.

(i) Prove that f does not have a weak derivative in any $L^p((-1,1))$.

(ii) Prove that instead there holds, for every $\varphi \in C_c^\infty((-1,1))$,

$$-\int_{-1}^{1} f(x)\varphi'(x)dx = \mathrm{p.\,v.}\int_{-1}^{1} \frac{\varphi(x)}{x}dx := \lim_{\varepsilon\to 0}\int_{(-1,1)\setminus[-\varepsilon,\varepsilon]} \frac{\varphi(x)}{x}dx,$$

The operator $\varphi \mapsto \mathrm{p.\,v.}\int_{-1}^{1} \frac{\varphi(x)}{x}dx$ is called *Cauchy principal value* of $1/x$.

(iii) Find an explicit expression for $\mathrm{p.\,v.}\int_{-1}^{1} \frac{\varphi(x)}{x}dx$ an absolutely convergent integral involving φ.

Remark. This exercise hints at the following fact: the weak derivative of $\log|x|$ is $\mathrm{p.\,v.}(1/x)$, which is not an ordinary function of x but rather a linear operator over the space of test functions. This heuristic consideration, familiar to every physicists, becomes rigorous and systematic in the *theory of distributions*.

Solution. (i) Suppose that f' existed in $L^1((-1,1))$. Then the integral $\int_{-1}^{1} f'(x)\varphi(x)dx$ had to be absolutely convergent for every $\varphi \in C_c^\infty((-1,1))$ and so we may estimate, for every $\varepsilon \in (0,1)$,

$$\int_{-1}^{1} |f'(x)\varphi(x)|dx \geq \int_{\varepsilon}^{1} |f'(x)||\varphi(x)|dx = \int_{\varepsilon}^{1} \frac{|\varphi(x)|}{x}dx,$$

but if $\varphi(0) \neq 0$ we get a contradiction by taking the limit as $\varepsilon \to 0$.

(ii) Since $f \in L^1(-1,1)$ and is differentiable away from the origin, we may integrate by parts as follows:

$$\begin{aligned}
&\int_{-1}^{1} f(x)\varphi'(x)dx \\
&= \lim_{\varepsilon\to 0}\int_{(-1,1)\setminus[-\varepsilon,\varepsilon]} \log|x|\varphi'(x)dx \\
&= \lim_{\varepsilon\to 0}\left[\int_{-1}^{-\varepsilon} \log|x|\varphi'(x)dx + \int_{\varepsilon}^{1} \log|x|\varphi'(x)dx\right] \\
&= \lim_{\varepsilon\to 0}\left[\varphi(-\varepsilon)\log\varepsilon - \int_{-1}^{-\varepsilon} \frac{\varphi(x)}{x}dx - \varphi(\varepsilon)\log\varepsilon - \int_{\varepsilon}^{1} \frac{\varphi(x)}{x}dx\right] \\
&= \lim_{\varepsilon\to 0}\Big[(\varphi(-\varepsilon) - \varphi(\varepsilon))\log\varepsilon\Big] - \lim_{\varepsilon\to 0}\int_{(-1,1)\setminus[-\varepsilon,\varepsilon]} \frac{\varphi(x)}{x}dx.
\end{aligned}$$

where we could separate the limits since the first one exists and in particular

$$\begin{aligned}\lim_{\varepsilon\to 0}\Big[(\varphi(-\varepsilon)-\varphi(\varepsilon))\log\varepsilon\Big] &= \lim_{\varepsilon\to 0}\Big[\frac{\varphi(-\varepsilon)-\varphi(\varepsilon)}{\varepsilon}\varepsilon\log\varepsilon\Big] \\ &= 2\varphi'(0)\lim_{\varepsilon\to 0}\Big[\varepsilon\log\varepsilon\Big] \\ &= 0.\end{aligned}$$

(iii) Arguing similarly as in (ii), with a change of variable we obtain

$$\begin{aligned}\text{p. v.}\int_{-1}^{1}\frac{\varphi(x)}{x}dx &= \lim_{\varepsilon\to 0}\left[\int_{-1}^{-\varepsilon}\frac{\varphi(x)}{x}dx+\int_{\varepsilon}^{1}\frac{\varphi(x)}{x}dx\right] \\ &= \lim_{\varepsilon\to 0}\left[-\int_{\varepsilon}^{1}\frac{\varphi(-x)}{x}dx+\int_{\varepsilon}^{1}\frac{\varphi(x)}{x}dx\right] \\ &= \lim_{\varepsilon\to 0}\left[\int_{\varepsilon}^{1}\frac{\varphi(x)-\varphi(-x)}{x}dx\right] \\ &= \int_{0}^{1}\frac{\varphi(x)-\varphi(-x)}{x}dx,\end{aligned}$$

where we could pass to the limit since, by the mean value theorem,

$$\left|\frac{\varphi(x)-\varphi(-x)}{x}\right| \le 2\|\varphi'\|_{L^\infty},$$

and so the integral was absolutely convergent. □

Exercise 3.3 Let $1 \le p \le \infty$. Recall from the lecture that a continuous linear extension operator $E\colon W^{1,p}(\mathbb{R}_+) \to W^{1,p}(\mathbb{R})$ can be constructed by even reflection across 0 (Satz 7.3.3).

Construct a linear operator $E\colon W^{2,p}(\mathbb{R}_+) \to W^{2,p}(\mathbb{R})$ satisfying:

- $(Eu)|_{\mathbb{R}_+} = u$ for every $u \in W^{2,p}(\mathbb{R}_+)$;
- $\|Eu\|_{W^{2,p}(\mathbb{R})} \le C\|u\|_{W^{2,p}(\mathbb{R}_+)}$ for a constant $C > 0$ independent of u.

Solution. We begin by defining operator of odd reflection

$$Fu(x) = \begin{cases} u(x) & \text{for } x > 0, \\ 2u(0) - u(-x) & \text{for } x < 0, \end{cases}$$

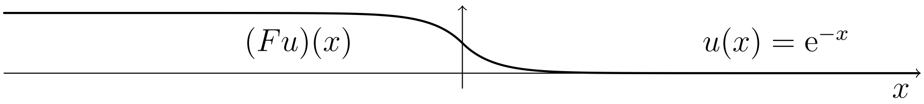


Figure 1: Extension by odd reflection.

We claim that $Fu \in W^{2,p}_{\mathrm{loc}}(\mathbb{R})$ with

$$u'(x) = \begin{cases} u'(x) & \text{for } x > 0, \\ u'(-x) & \text{for } x < 0, \end{cases} \quad \text{and} \quad u''(x) = \begin{cases} u''(x) & \text{for } x > 0, \\ -u''(-x) & \text{for } x < 0, \end{cases}$$

Indeed, call g and h the two functions defined by the expressions above and let $\varphi \in C^\infty_c(\mathbb{R})$ be arbitrary. Then, using integration by parts for Sobolev functions (Lemma 7.3.0 of the lectures) and the fundamental theorem of calculus for Sobolev functions (Satz 7.3.1), we obtain

$$\begin{aligned} -\int_{\mathbb{R}} (Fu)\varphi'\,dx &= -\int_{-\infty}^{0} 2u(0)\varphi'(x) - u(-x)\varphi'(x)\,dx - \int_0^\infty u(x)\varphi'(x)\,dx \\ &= -2u(0)\varphi(0) + \int_{-\infty}^{0} u(-x)\varphi'(x)\,dx - \int_0^\infty u(x)\varphi'(x)\,dx \\ &= -\int_{-\infty}^{0} -u'(-x)\varphi(x)\,dx + \int_0^\infty u'(x)\varphi(x)\,dx = \int_{\mathbb{R}} g\varphi\,dx, \end{aligned}$$

which proves that g is the first weak derivative of Fu. Similarly, h is the weak derivative of g.

Although $(Fu)', (Fu)'' \in L^p(\mathbb{R})$, the same does not hold for Fu unless $u(0) = 0$, and so we let $\psi \in C^\infty(\mathbb{R})$ be a fixed smooth cut-off function identically equal to 1 on $\mathbb{R}_+$ and 0 on $(-\infty, -1)$ and define the linear operator

$$Eu(x) = \psi(x)(Fu)(x), \quad x \in \mathbb{R}.$$

Recalling that

$$|u(0)| \le \|u\|_{L^\infty(\mathbb{R}_+)} \le C\|u\|_{W^{1,p}(\mathbb{R}_+)},$$

we may estimate with the Leibniz rule

$$\begin{aligned} \|Eu\|_{L^p(\mathbb{R})} &\le C\Big(|u(0)| + \|u\|_{L^p(\mathbb{R}_+)}\Big) \le C\|u\|_{W^{1,p}(\mathbb{R}_+)}, \\ \|(Eu)'\|_{L^p(\mathbb{R})} &\le C\|u\|_{W^{1,p}(\mathbb{R}_+)}, \\ \|(Eu)''\|_{L^p(\mathbb{R})} &\le C\|u\|_{W^{2,p}(\mathbb{R}_+)}, \end{aligned}$$

where we have included in C the dependence on the C^2 norm of ψ. So the linear operator E above constructed is the required one. $\square$

Exercise 3.4 *Note:* this exercise is supplementary to the previous one. Solve that first! The solution you find may differ from the procedure here described.

(i) Let $k \in \mathbb{N}$. Show that there exist $a_1, \dots, a_k \in \mathbb{R}$ such that for any polynomial $p\colon \mathbb{R} \to \mathbb{R}$, $p(x) = \sum_{\ell=0}^{k-1} p_\ell x^\ell$ of degree $k-1$ and every $x < 0$, there holds

$$\sum_{j=1}^{k} a_j\, p\Big(\frac{-x}{j}\Big) = p(x).$$

(ii) Let $1 \le p \le \infty$ and $k \in \mathbb{N}$. Let $a_1, \dots, a_k \in \mathbb{R}$ as in (i). Prove that the map

$$E\colon u \mapsto Eu, \qquad (Eu)(x) := \begin{cases} u(x) & \text{for } x > 0, \\ \sum_{j=1}^{k} a_j u\Big(\frac{-x}{j}\Big) & \text{for } x < 0, \end{cases}$$

defines a linear operator $E\colon W^{k,p}(\mathbb{R}_+) \to W^{k,p}(\mathbb{R})$ so that for every $u \in W^{k,p}(\mathbb{R}_+)$ and any integer $0 \le \alpha \le k$

$$\|D^\alpha(Eu)\|_{L^p(\mathbb{R})} \le C \|D^\alpha u\|_{L^p(\mathbb{R}_+)},$$

for a constant $C > 0$ independent of u.

Solution. (i) Let $k \in \mathbb{N}$. For $m \in \{0, \dots, k-1\}$ and $p(x) = x^m$, we obtain the equation

$$\forall x \in \mathbb{R} \quad \sum_{j=1}^{k} a_j \Big(\frac{-x}{j}\Big)^m = x^m \qquad \iff \qquad \sum_{j=1}^{k} \frac{a_j}{j^m} = (-1)^m.$$

Equivalently,

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{k} \\ 1 & (\frac{1}{2})^2 & (\frac{1}{3})^2 & \dots & (\frac{1}{k})^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (\frac{1}{2})^{k-1} & (\frac{1}{3})^{k-1} & \dots & (\frac{1}{k})^{k-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_k \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ \vdots \\ (-1)^{k-1} \end{pmatrix}.$$

The matrix A on the left hand side is a Vandermonde matrix. In particular,

$$\det A = \prod_{1 \le i < j \le k} \Big(\frac{1}{j} - \frac{1}{i}\Big) \ne 0$$

which implies that a unique solution $(a_1, \dots, a_k) \in \mathbb{R}^k$ to the linear system exists. By linearity,

$$\sum_{j=1}^{k} a_j\, p\Big(\frac{-x}{j}\Big) = p(x).$$

holds not only for monomials $p(x) = x^m$ with $m \in \{0, \dots, k-1\}$ but in fact for arbitrary polynomials of degree $k-1$.

(ii) Let $k \in \mathbb{N}$ be fixed and $a_1, \dots, a_k$ as in part (i). Given $u \in W^{k,p}(\mathbb{R}_+)$, consider (Eu) as given on the exercise; note that $(Eu) \in L^p(\mathbb{R})$ since $u \in L^p(\mathbb{R}_+)$ We are going to prove that g_α $(0 \leq \alpha \leq k)$ given by

$$g_\alpha(x) := \begin{cases} D^\alpha u(x) & \text{for } x > 0, \\ \sum_{j=1}^{k} \Big(-\frac{1}{j}\Big)^\alpha a_j (D^\alpha u)\Big(\frac{-x}{j}\Big) & \text{for } x < 0 \end{cases}$$

is the α-th weak derivative of (Eu). First of all, $g_\alpha \in L^p(\mathbb{R})$ since $(D^\alpha u) \in L^p(\mathbb{R}_+)$.

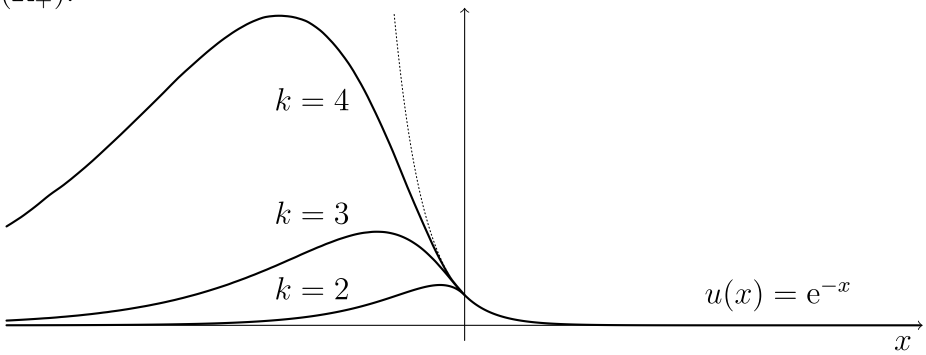


Figure 2: Extensions $(Eu)(x)$ of $u(x) = \mathrm{e}^{-x}$ for $k = 2, 3, 4$.

For $\alpha = 0$ we have $g_0 = Eu$ by construction. Suppose $D^\alpha(Eu) = g_\alpha$ for some $\alpha < k$. For $\varphi \in C_c^\infty(\mathbb{R})$, using Lemma 7.3.0 and Satz 7.3.1, we get

$$\begin{aligned}
&(-1)^{\alpha+1} \int_{\mathbb{R}} (Eu) D^{\alpha+1}\varphi\, dx = -\int_{\mathbb{R}} D^\alpha(Eu)\varphi'\, dx = -\int_{\mathbb{R}} g_\alpha \varphi'\, dx \\
&= -\sum_{j=1}^{k} \Big(-\tfrac{1}{j}\Big)^\alpha a_j \int_{-\infty}^{0} (D^\alpha u)\Big(-\tfrac{x}{j}\Big)\varphi'(x)\, dx - \int_0^\infty (D^\alpha u)\varphi'\, dx \\
&= \sum_{j=1}^{k} \Big(-\tfrac{1}{j}\Big)^{\alpha+1} a_j \int_{-\infty}^{0} (D^{\alpha+1} u)\Big(-\tfrac{x}{j}\Big)\varphi(x)\, dx - \sum_{j=1}^{k} \Big(-\tfrac{1}{j}\Big)^\alpha a_j (D^\alpha u)(0)\,\varphi(0) \\
&\quad + \int_0^\infty (D^{\alpha+1} u)\varphi\, dx + (D^\alpha u)(0)\,\varphi(0) \\
&= \int_{\mathbb{R}} g_{\alpha+1}\varphi\, dx + \Big(1 - \sum_{j=1}^{k} \Big(-\tfrac{1}{j}\Big)^\alpha a_j\Big)(D^\alpha u)(0)\,\varphi(0).
\end{aligned}$$

Since $\sum_{j=1}^{k}\left(-\frac{1}{j}\right)^{\alpha} a_j = 1$ was proven in part (i) (set $x = 1$ and $m = \alpha$), the claim $D^{\alpha+1}(Eu) = g_{\alpha+1}$ follows. Hence, $E(W^{k,p}(\mathbb{R}_+)) \subset W^{k,p}(\mathbb{R})$ and, for any integer $0 \leq \alpha \leq k$,

$$\begin{aligned}
\|D^{\alpha}(Eu)\|_{L^p(\mathbb{R})} &\leq \|D^{\alpha}u\|_{L^p(\mathbb{R}_+)} + \left\|\sum_{j=1}^{k}\left(-\tfrac{1}{j}\right)^{\alpha} a_j (D^{\alpha}u)\left(\tfrac{\cdot}{j}\right)\right\|_{L^p(\mathbb{R}_+)} \\
&\leq \|D^{\alpha}u\|_{L^p(\mathbb{R}_+)} + \sum_{j=1}^{k}\frac{|a_j|}{j^{\alpha}} j^{\frac{1}{p}} \|D^{\alpha}u\|_{L^p(\mathbb{R}_+)} \\
&\leq C_{k,p}\|D^{\alpha}u\|_{L^p(\mathbb{R}_+)}.
\end{aligned}$$

□

Hints to Exercises.

3.1 Recall that L^p spaces are reflexive for $p \in (1, \infty)$ and that L^∞ is the dual of the separable space L^1.

3.2 write $\int_{-1}^{1} f(x)\varphi'(x)dx = \lim_{\varepsilon\to 0} \int_{(-1,1)\setminus[-\varepsilon,\varepsilon]} f(x)\varphi'(x)dx$, use that f is smooth away from 0 and integrate by parts.

3.3 Argue carefully by odd reflection, and then use cut-off functions.