Exercise 4.1 Let $I :=]a, b[$ for $-\infty < a < b < \infty$. Given $f \in C^0(\overline{I})$, consider the boundary value problem

$$\begin{cases} -v'' + v = f & \text{in } I, \\ \quad v(a) = 0, & \\ \quad v(b) = 0, & \end{cases} \tag{$*$}$$

(i) Show that $(*)$ has a weak solution $u \in H_0^1(I)$, that is, satisfying

$$\int_I u'\varphi'\,dx + \int_I u\varphi\,dx = \int_I f\varphi\,dx$$

for every $\varphi \in H_0^1(I)$, and that it is unique.

(ii) Prove that the weak solution u from (i) is in fact a classical solution $u \in C^2(\overline{I})$.

(iii) Prove that the boundary-value problem

$$\begin{cases} -v'' + v = g & \text{in } I, \\ \quad v(a) = \alpha, & \\ \quad v(b) = \beta, & \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$ and $g \in C^0(\overline{I})$ has unique classical solution $v \in C^2(\overline{I})$.

Solution. (i) Recall that the space $H_0^1(I)$ is a closed subspace of the Hilbert space $H^1(I)$ and thus is itself Hilbert.

Given $f \in C^0(\overline{I})$, the map

$$\ell_f \colon H_0^1(I) \to \mathbb{R}, \qquad \ell_f(\varphi) := \int_I f(x)\varphi(x)\,dx$$

is a linear, continuous functional, with

$$|\ell_f(\varphi)| \le \|f\|_{L^2}\|\varphi\|_{L^2} \le \|f\|_{L^2}\|\varphi\|_{H^1}.$$

By the Riesz representation Theorem, there exists a unique $u \in H_0^1(I)$ satisfying

$$\int_I f\varphi\,dx =: \ell_f(\varphi) = (u, \varphi)_{H^1} = \int_I u'\varphi'\,dx + \int_I u\varphi\,dx$$

for every $\varphi \in H_0^1(I)$.

(ii) Let $u \in H_0^1(I)$ be the weak solution to the equation $-u'' + u = f$ in I found in part (i). We have in particular

$$\forall \varphi \in C_c^\infty(I): \quad -\int_I u'\varphi'\,dx = \int_I (u-f)\varphi\,dx.$$

Hence, the function $u' \in L^2(I)$ has the weak derivative $(u-f) \in L^2(I)$ and we conclude $u' \in H^1(I)$. Therefore, u' has a continuous representative satisfying

$$u'(x) = u'(a) + \int_a^x (u-f)(t)\,dt. \tag{1}$$

Since also $f \in C^0(\overline{I})$, the right hand side is then in $C^1(\overline{I})$ and this implies $u \in C^2(\overline{I})$.

(iii) We let $v_0 \in C^\infty(\overline{I})$ be given by

$$v_0(x) = \alpha + \frac{x-a}{b-a}(\beta - \alpha).$$

and let $f = g - v_0 \in C^0(\overline{I})$. For this f we let $u \in C^2(\overline{I})$ be the solution obtained in (ii). Then $v := u + v_0 \in C^2(\overline{I})$ solves the problem, since

$$\begin{cases} -v'' + v = -u'' - v_0'' + u + v_0 = -u'' + u + v_0 = f + v_0 = g, \\ \quad v(a) = u(a) + u_0(a) = u_0(a) = \alpha, \\ \quad v(b) = u(b) + u_0(b) = u_0(b) = \beta. \end{cases}$$

To prove uniqueness, let $\tilde{v} \in C^2(\overline{I})$ be another solution to the boundary value problem. Then the function $u := v - \tilde{v} \in C^2(\overline{I})$ satisfies $-u'' + u = 0$ with $u(a) = 0 = u(b)$, so (for instance) with integration by parts we see

$$\int_I u^2\,dx = \int_I u''u\,dx = -\int_I |u'|^2\,dx \leq 0$$

which implies $u \equiv 0$ and hence $\tilde{v} \equiv v$. □

Exercise 4.2 Let $I :=]a,b[$ for $-\infty < a < b < \infty$. Let $g \in C^1(\overline{I})$ and $h, f \in C^0(\overline{I})$. Assume that $g(x) \geq \lambda > 0$ and $h(x) \geq 0$ for every $x \in \overline{I}$ and consider the boundary value problem

$$\begin{cases} -(g\,u')' + h\,u = f \quad \text{in } I, \\ \qquad u(a) = 0, \\ \qquad u(b) = 0. \end{cases} \tag{†}$$

(i) Apply the Riesz representation theorem in a suitable Hilbert space to prove that (†) has a unique weak solution $u \in H^1_0(I)$.

(ii) Prove that the weak solution u from (i) is in fact a classical solution $u \in C^2(\overline{I})$.

Solution. (i) Define the new scalar product

$$\langle u, v\rangle := \int_I (g\, u'v' + h\, uv)\, dx$$

for all $u, v \in H^1_0(I)$. By assumption,

$$\langle u, u\rangle = \int_I (g\,|u'|^2 + h\,|u|^2)\, dx \geq \lambda \int_I |u'|^2\, dx$$

for any $u \in H^1_0(I)$. Moreover, using Poincaré's inequality,

$$\begin{aligned}\langle u, u\rangle &\leq \|g\|_{C^0} \int_I |u'|^2\, dx + \|h\|_{C^0} \int_I |u|^2\, dx \\ &\leq \Big(\|g\|_{C^0} + (b-a)^2 \|h\|_{C^0}\Big) \int_I |u'|^2\, dx.\end{aligned}$$

Hence, $\langle\cdot,\cdot\rangle$ is equivalent to the standard scalar product $(u,v)_{H^1_0}$ on $H^1_0(I)$, and in particular $(H^1_0(I), \langle\cdot,\cdot\rangle)$ is Hilbert. Given $f \in C^0(\overline{I})$, the map

$$\ell_f \colon H^1_0(I) \to \mathbb{R}, \qquad \ell_f(\varphi) := \int_I f(x)\varphi(x)\, dx$$

is a linear, continuous functional:

$$|\ell_f(\varphi)| \leq \|f\|_{L^2} \|\varphi\|_{L^2} \leq (b-a)\|f\|_{L^2}\|\varphi\|_{H^1_0}.$$

By the Riesz representation Theorem, there exists a unique $u \in H^1_0(I)$ satisfying

$$\int_I f\varphi\, dx = \int_I g\, u'\varphi' + h\, u\varphi\, dx$$

for every $\varphi \in H^1_0(I)$. which is equivalent to being a weak solution of the equation (†), i.e.

$$-\int_I gu'\varphi'\, dx = \int_I (hu - f)\varphi\, dx.$$

for every $\varphi \in H^1_0(I)$. Hence, the function gu' has the weak derivative $(hu - f) \in L^2(I)$ and we conclude $gu' \in H^1(I)$. Therefore, gu' has a continuous representative satisfying

$$(gu')(x) = (gu')(a) + \int_a^x (hu - f)(t)\, dt.$$

Since $h, f \in C^0(\overline{I})$, the right-hand side is in $C^1(\overline{I})$. Finally, $gu' \in C^1(\overline{I})$ and $0 < \lambda \leq g \in C^1(\overline{I})$ imply $u' \in C^1(\overline{I})$. Hence, $u \in C^2(\overline{I})$ as claimed. □

Exercise 4.3 Let $1 \leq p \leq \infty$, $I =]a, b[$ for $-\infty < a < b < \infty$ and $u \in W^{1,p}(I)$.

(i) Let $G \in C^1(\mathbb{R})$. Prove that $G \circ u$ is in $W^{1,p}(I)$ and that the chain rule holds for weak derivatives:

$$(G \circ u)' = (G' \circ u)u'.$$

(ii) Prove that $|u| \in W^{1,p}(I)$ and compute its weak derivative.

Solution. (i) Assume first $p < \infty$. From the bound

$$\|u\|_{L^\infty(I)} \leq C\|u\|_{W^{1,p}(I)},$$

we deduce that $G \circ u$ is also bounded and in particular in $L^p(I)$.

To compute its weak derivative we argue by density. Let $(u_k)_{k\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R})$ be a sequence of smooth functions with $u_k \to u$ in $W^{1,p}(I)$; note in particular that, because of the above inequality, we have that

$$u_n \to u \quad \text{uniformly on } \overline{I};$$

In particular there exists some $M > 0$ so that

$$\sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty(I)} \leq M.$$

For $\varphi \in C_c^\infty(I)$ we then have

$$\int_I (G \circ u_n)(x)\varphi'(x)dx = -\int_I (G' \circ u_n)(x)u_n'(x)\varphi(x)dx. \tag{$\circ$}$$

Now, on the one hand we have

$$|(G \circ u_n)(x)\varphi'(x)| \leq \Big(\sup_{t\in[-M,M]} |G(t)|\Big)|\varphi'(x)| \quad \forall x \in I,$$

and so the expression on the right-hand side is bounded in x uniformly in n and pointwise convergent.

On the other hand we have $\forall x \in I$

$$\begin{aligned}|(G' \circ u_n)(x)u_n'(x)\varphi(x)| &\leq \Big(\sup_{t\in[-M,M]} |G'(t)|\Big)|\varphi(x)||u_n'(x)| \\ &\leq C|u'(x)| + C|u_n'(x) - u'(x)|,\end{aligned}$$

and so the expression on the right-hand side is, up to an error $o(1) \to 0$, in $L^p(I)$ bounded by the fixed function $C|u'(x)| \in L^p(I)$, uniformly in n.

By the dominated convergence theorem we may then pass to the limit as $n \to \infty$ in ($\circ$) and deduce that $G \circ u$ has the weak derivative $(G' \circ u)u' \in L^p(I)$, as claimed.

The case $p = \infty$ may now be dealt with as in part (ii) of the proof of Corollary 7.3.2 in the notes.

(ii) Clearly $|u| \in L^p(I)$. We now let $G(s) = |s|$ for $s \in \mathbb{R}$ and consider, for $\varepsilon > 0$, the function

$$G_\varepsilon(x) = \sqrt{x^2 + \varepsilon^2}$$

which is $C^1(I)$ and converges uniformly to F as $\varepsilon \to 0$. Moreover we have

$$G'_\varepsilon(x) = \frac{x}{\sqrt{x^2+\varepsilon^2}} \longrightarrow \operatorname{sign}(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0, \end{cases}$$

pointwise for $x \neq 0$ as $\varepsilon \to 0$. By (i) we then have, for fixed $\varphi \in C_c^\infty(I)$,

$$\int_I (G_\varepsilon \circ u)(x)\varphi'(x)dx = -\int_I (G'_\varepsilon \circ u)(x)u'(x)\varphi(x)dx. \tag{$\star$}$$

Now we notice that

$$(G'_\varepsilon \circ u)(x)u'(x)\varphi(x) \longrightarrow \operatorname{sign}(u(x))\, u'(x)\varphi(x) \quad \text{a.e. in } I$$

and moreover, for every $\varepsilon > 0$,

$$|(G'_\varepsilon \circ u)(x)u'(x)\varphi(x)| \leq |u'(x)\varphi(x)|,$$

and the right-hand side is in $L^p(I)$. With the dominated convergence theorem we may then pass to the limit in ($\star$) and deduce that $|u| \in W^{1,p}(I)$ with weak derivative given by $|u|' = \operatorname{sign}(u)u'$. □

Exercise 4.4 (Euler's Paradox) Consider the problem of minimizing the functional

$$\mathscr{F}(u) = \int_0^1 (u'(t)^2 - 1)^2 dt$$

among functions $u : [0,1] \to \mathbb{R}$ subject to the boundary condition:

$$u(0) = u(1) = 0.$$

(i) Prove that every $u \in C^1([0,1])$ solving the above minimization problem must solve the boundary value problem

$$\begin{cases} u'(t)(u'(t)^2 - 1) = c & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{$\bowtie$}$$

for some $c \in \mathbb{R}$.

(ii) Find a classical solution to ($\bowtie$) for a c of your choice. Is it also a solution to the minimization problem for $\mathscr{F}$?

(iii) Find a weak solution $u \in W^{1,\infty}((0,1))$ solving both ($\bowtie$) and the minimization problem. Is this solution unique?

(iv) Compute the value

$$\inf \Big\{ \mathscr{F}(u) : u \in C^1((0,1)), u(0) = 0 = u(1) \Big\}.$$

Is this value attained by some $u \in C^1((0,1))$?

Solution. (i) If u minimizes $\mathscr{F}$ with the given boundary condition, it must be true that

$$\frac{d}{d\varepsilon}\mathscr{F}(u + \varepsilon\varphi)\bigg|_{\varepsilon=0} = 0 \quad \text{for every } \varphi \in C_c^\infty((0,1)).$$

From

$$\begin{aligned} \frac{d}{d\varepsilon}\mathscr{F}(u+\varepsilon\varphi) &= \int_0^1 \frac{d}{d\varepsilon}\Big((u'(t)+\varepsilon\varphi'(t))^2 - 1\Big)^2 dt \\ &= \int_0^1 4\Big((u'(t)+\varepsilon\varphi'(t))^2 - 1\Big)(u'(t)+\varepsilon\varphi'(t))\varphi'(t)dt \end{aligned}$$

we have

$$\frac{d}{d\varepsilon}\mathscr{F}(u + \varepsilon\varphi)\bigg|_{\varepsilon=0} = 4\int_0^1 (u'(t)^2 - 1)u'(t)\varphi'(t)dt,$$

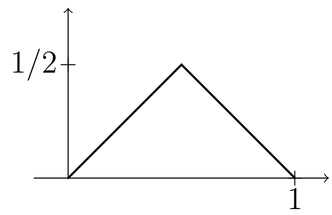
and thus ($\bowtie$) holds by the du Bois-Reymond lemma.

(ii) Perhaps the easiest choice is $c = 0$ so that a C^1 solution ($\bowtie$) is $u \equiv 0$. It is not a solution to the minimization problem: since $\mathscr{F}(0) = 1$ we may construct explicitly another function u with $\mathscr{F}(u) < 1$ (just pick one of the u_n's constructed in (iv)).

No matter what choice is made, it is not a solution to the minimization problem; we shall prove in (iv) that there is no solution of class C^1 to the problem.

(iii) Inspecting $\mathscr{F}$ we infer that, for any $u \in W^{1,\infty}((0,1))$, $\mathcal{F}(u) \geq 0$ and $\mathcal{F}(u) = 0$ if and only if $|u'(t)| = 1$ a.e. A simple choice falls then upon

$$u(t) = \begin{cases} t & \text{for } t \in [0, 1/2], \\ 1/2 - t & \text{for } t \in [1/2, 1], \end{cases}$$



To see that u is actually a weak solution of ($\bowtie$), namely that

$$\int_0^1 u'(t)(u'(t)^2 - 1)\varphi'(t)dt = 0 \quad \forall \varphi \in C_c^\infty((0,1)),$$

we notice that the weak derivative of u is

$$u'(t) = \begin{cases} 1 & \text{for } t \in (0, 1/2), \\ -1 & \text{for } t \in (1/2, 1), \end{cases}$$

and thus, for every $\varphi \in C_c^\infty((0,1))$

$$\begin{aligned} &\int_0^1 u'(t)(u'(t)^2 - 1)\varphi'(t)dt \\ &= \int_0^{1/2} u'(t)(u'(t)^2 - 1)\varphi'(t)dt + \int_{1/2}^1 u'(t)(u'(t)^2 - 1)\varphi'(t)dt \\ &= 0, \end{aligned}$$

since both integrals vanish. Hence ($\bowtie$) is satisfied by the du Bois-Reymond lemma.

The choice of u is not unique: we can for instance modify u by adding more spikes, always keeping the slope equal to ± 1, as indicated in the Figure 1.

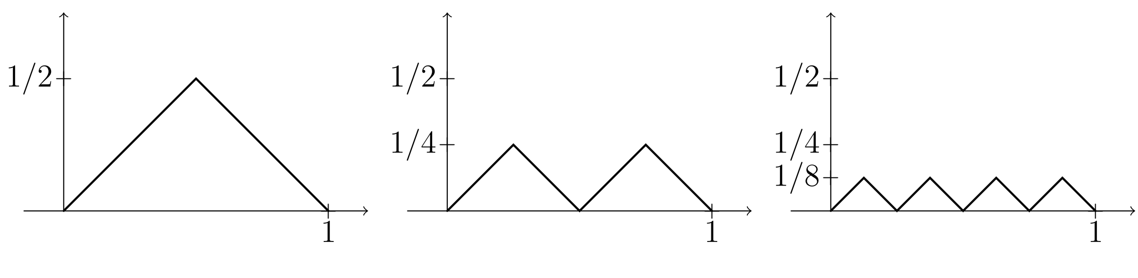


Figure 1: Infinitely many minimizers.

The process of verifying that every piecewise linear function above is also a weak solution to ($\bowtie$) is analogous to that for u. The possible choices are infinite.

(iv) We prove that

$$\inf\left\{\mathscr{F}(u) : u \in C^1((0,1)), u(0) = 0 = u(1)\right\} = 0$$

by constructing a sequence $(u_n)_n \subset C^1([0,1])$ with zero boundary value so that $\mathscr{F}(u_n) \to 0$ as $n \to \infty$. For $n \geq 4$, we define

$$u_n(t) = \begin{cases} t & \text{for } t \in [0, 1/2 - 1/n], \\ f_n(t) & \text{for } t \in [1/2 - 1/n, 1/2 + 1/n], \\ 1/2 - t & \text{for } t \in [1/2 + 1/n, 1], \end{cases}$$

where $f_n(t)$ is the parabola passing through the points $(1/2 - 1/n, 1/2 - 1/n)$ $(1/2 + 1/n, 1/2 - 1/n)$ and so that u_n is of class C^1, as shown in Figure 2 (one may even improve the construction so that the u_n's are smooth). Then for this sequence one has

$$\mathscr{F}(u_n) = o(1) \quad \text{as } n \to \infty.$$

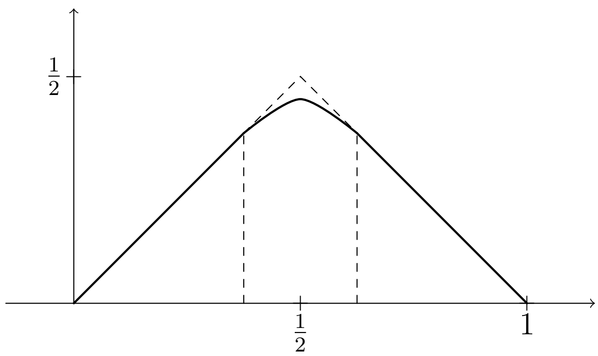


Figure 2: Construction of u_n.

However, no $u \in C^1([0,1])$ with $u(0) = 0 = u(1)$ attains the minimum. If this were true, by the computation above it had to be $\mathscr{F}(u) = 0$. However, this can happen if and only if $|u'(t)| \equiv 1$. On the other hand by Rolle's theorem there also exists $\xi \in (0,1)$ where $u'(\xi) = 0$, contradiction. $\square$

Hints to Exercises.

4.1 For (i), use the Riesz representation theorem. For (iii), construct first directly a function satisfying the boundary conditions and then resort to (i)-(ii).

4.2 Apply Riesz to a suitable Hilbert space, and argue similarly as in 4.1.

4.3 Argue by approximation on u for (i) and on $x \mapsto |x|$ for (ii).

4.4 For (ii), notice that $\mathscr{F}(u) \geq 0$, and $\mathscr{F}(u) = 0$ if and only if...