Exercise 5.1 For $p \in [1, \infty]$, the space of *periodic Sobolev functions* $W^{1,p}_{\text{per}}((0, 2\pi))$ is the subset of functions $\varphi \in W^{1,p}((0, 2\pi))$ so that

$$\int_0^{2\pi} \varphi'(x)\psi(x)dx = -\int_0^{2\pi} \varphi(x)\psi'(x)dx$$

for every smooth function $\psi \in C^\infty([0, 2\pi])$ so that $\psi^{(k)}(0) = \psi^{(k)}(2\pi)$ for every $k \in \mathbb{N}$. A similar definition is given for $W^{k,p}_{\text{per}}((0, 2\pi))$.

Recall that, for a periodic function $\varphi : (0, 2\pi) \to \mathbb{R}$, its Fourier coefficients are

$$\widehat{\varphi}(n) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(x)\mathrm{e}^{-inx}dx, \quad n \in \mathbb{N},$$

and its Fourier series is

$$FS(\varphi)(x) = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(n)\mathrm{e}^{inx}, \quad x \in [0, 2\pi].$$

Fact: For $\varphi \in L^2((0, 2\pi))$, $FS(\varphi)$ converges to φ in L^2-norm.

(i) Prove that $\varphi \in W^{1,2}_{\text{per}}((0, 2\pi))$ if and only if

$$\sum_{n\in\mathbb{Z}} (1 + n^2)|\widehat{\varphi}(n)|^2 < \infty,$$

(ii) Prove Sobolev embedding for periodic functions using only (i) and the "Fact" above, that is, show that if $\varphi \in W^{1,2}_{\text{per}}((0, 2\pi))$, then φ can be identified with a function in $C^0[0, 2\pi]$ so that $\varphi(0) = \varphi(2\pi)$ and

$$\|\varphi\|_{C^0((0,2\pi))} \le C\|\varphi\|_{W^{1,2}((0,2\pi))}.$$

(iii) Argue similarly as in (i) and prove functions $\varphi \in W^{k,2}_{\text{per}}((0, 2\pi))$ are exactly those so that

$$\sum_{n\in\mathbb{Z}} (1 + n^{2k})|\widehat{\varphi}(n)|^2 < \infty.$$

Solution. (i) From Parseval's identity we have

$$\|\varphi\|^2_{W^{1,2}((0,2\pi))} = \|\varphi\|^2_{L^2((0,2\pi))} + \|\varphi'\|^2_{L^2((0,2\pi))} = \frac{1}{2\pi}\left(\sum_{n\in\mathbb{Z}} |\widehat{\varphi}(n)|^2 + \sum_{n\in\mathbb{Z}} |\widehat{\varphi'}(n)|^2\right).$$

Since $x \mapsto \mathrm{e}^{inx}$ is smooth and periodic, for the Fourier coefficients of φ' we have, integrating by parts

$$\widehat{\varphi'}(n) = \frac{1}{2\pi}\int_0^{2\pi} \varphi'(x)\mathrm{e}^{-inx}dx = \frac{in}{2\pi}\int_0^{2\pi} \varphi(x)\mathrm{e}^{-inx}dx,$$

and so $\widehat{\varphi'}(n) = in\widehat{\varphi}(n)$. Consequently

$$\|\varphi\|^2_{W^{1,2}((0,2\pi))} = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}(1+n^2)|\widehat{\varphi}(n)|^2,$$

and thus φ is in $W^{1,2}_{\mathrm{per}}$ if and only if the sum in question is finite.

(ii) We prove that the Fourier series of φ is uniformly convergent: since it converges in L^2 to φ, by the uniqueness of the limit then the convergence must be also uniform and so φ will be (identified with) a continuous function. By the Cauchy-Schwarz inequality over ℓ^2, we see that, for any $N \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=-N}^{N} |\widehat{\varphi}(n)| &= \sum_{n=-N}^{N} \frac{\sqrt{1+n^2}}{\sqrt{1+n^2}}|\widehat{\varphi}(n)| \\ &\le \left(\sum_{n=-N}^{N}\frac{1}{1+n^2}\right)\left(\sum_{n=-N}^{N}(1+n^2)|\widehat{\varphi}(n)|^2\right) \\ &\le C\sum_{n\in\mathbb{Z}}(1+n^2)|\widehat{\varphi}(n)|^2 \\ &\le C\|\varphi\|_{W^{1,2}((0,2\pi))},\end{aligned}$$

and this implies the uniform convergence of $SF(\varphi)$, in particular

$$\varphi(x) = \sum_{n\in\mathbb{Z}}\widehat{\varphi}(n)\mathrm{e}^{inx} \quad \forall x \in [0,2\pi].$$

Clearly then $\varphi(0) = \varphi(2\pi)$ and also $\|\varphi\|_{C^0} \le C\|\varphi\|_{W^{1,2}}$ by the triangle inequality and the fact that $|\mathrm{e}^{inx}| \equiv 1$.

(iii) Similarly as in (i), one has

$$\widehat{\varphi^{(k)}}(n) = (in)^k\widehat{\varphi}(n),$$

hence one sees that the $W^{k,2}$-norm squared of φ is equivalent to

$$\sum_{n\in\mathbb{Z}}(1+n^2+\cdots+n^{2k})|\widehat{\varphi}(n)|^2,$$

and since in turn we can always estimate

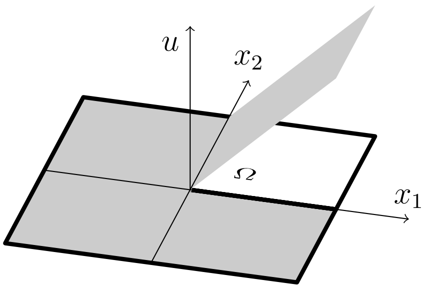
$$(1+n^{2k}) \le (1+n^2+\cdots+n^{2k}) \le C_k(1+n^{2k}),$$

for a suitable constant $C_k > 0$ depending only on k, the thesis follows. $\square$

Exercise 5.2 Find an open set $\Omega \subset \mathbb{R}^2$ and a function $u \in W^{1,\infty}(\Omega)$ which is not Lipschitz continuous.

Solution. Let $\Omega = (]-1,1[\times]-1,1[) \setminus ([0,1[\times\{0\})$ and let $u\colon \Omega \to \mathbb{R}$ be given by

$$u(x_1,x_2) := \begin{cases} 0 & \text{if } -1 < x_1 \le 0 \text{ or } x_2 < 0, \\ x_1 & \text{if } x_1 > 0 \text{ and } x_2 > 0. \end{cases}$$



Then, $\Omega \subset \mathbb{R}^2$ is open and u is bounded. For any $\varphi \in C_c^\infty(\Omega)$, we have

$$\begin{aligned} -\int_\Omega u \frac{\partial \varphi}{\partial x_1}\,dx &= -\int_0^1 \Big(\int_0^1 x_1 \frac{\partial \varphi}{\partial x_1}\,dx_1\Big)\,dx_2 \\ &= \int_0^1 \Big(\Big(\int_0^1 \varphi\,dx_1\Big) - x_1\varphi(x_1,x_2)\Big|_{x_1=0}^{x_1=1}\Big)\,dx_2 \\ &= \int_0^1 \int_0^1 \varphi\,dx_1\,dx_2, \end{aligned}$$

and

$$\begin{aligned} -\int_\Omega u \frac{\partial \varphi}{\partial x_2}\,dx &= -\int_0^1 \Big(\int_0^1 x_1 \frac{\partial \varphi}{\partial x_2}\,dx_2\Big)\,dx_1 \\ &= \int_0^1 \Big(0 - x_1\varphi(x_1,x_2)\Big|_{x_2=0}^{x_2=1}\Big)\,dx_1 = 0, \end{aligned}$$

where we used that $(1,x_2),(x_1,1) \in \partial\Omega$ for any $x_1,x_2 \in\,]-1,1[$ and $(x_1,0) \in \partial\Omega$ for $x_1 > 0$ which implies that φ vanishes at these points. Hence, the weak derivatives $\frac{\partial u}{\partial x_1} = \chi_{]0,1[^2} \in L^\infty(\Omega)$ and $\frac{\partial u}{\partial x_2} = 0 \in L^\infty(\Omega)$ exist and $u \in W^{1,\infty}(\Omega)$. However, since

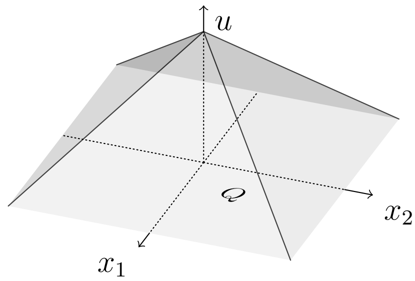
$$\frac{|u(\frac{1}{2},-\frac{1}{k}) - u(\frac{1}{2},\frac{1}{k})|}{|(\frac{1}{2},-\frac{1}{k}) - (\frac{1}{2},\frac{1}{k})|} = \frac{\frac{1}{2}}{\frac{2}{k}} = \frac{k}{4}$$

is well-defined for any $k > 1$ and unbounded for $k \to \infty$, we conclude that u is not Lipschitz continuous.

Remark. There are many more kinds of examples. The one we showed may very much differ from the one you found. $\square$

Exercise 5.3 (A tent for Rudolf L.) Let $Q = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| < 1,\ |x_2| < 1\}$. Let $u\colon Q \to \mathbb{R}$ be given by

$$u(x_1, x_2) = \begin{cases} 1 - x_1, & \text{if } x_1 > 0 \text{ and } |x_2| < x_1, \\ 1 + x_1, & \text{if } x_1 < 0 \text{ and } |x_2| < -x_1, \\ 1 - x_2, & \text{if } x_2 > 0 \text{ and } |x_1| < x_2, \\ 1 + x_2, & \text{if } x_2 < 0 \text{ and } |x_1| < -x_2. \end{cases}$$



For which exponents $1 \le p \le \infty$ is $u \in W^{1,p}(Q)$?

Solution. The function $u\colon Q \to \mathbb{R}$ is given by $u(x_1, x_2) = 1 - \max\{|x_1|, |x_2|\}$ and it is bounded in Q. Let $x = (x_1, x_2), y = (y_1, y_2) \in Q$ be arbitrary; w.l.o.g. $u(y) > u(x)$. Then

$$\begin{aligned} u(y) - u(x) &= \max\{|x_1|, |x_2|\} - \max\{|y_1|, |y_2|\} \\ &\le \begin{cases} |x_1| - |y_1| \le |x_1 - y_1| & \text{if } |x_1| \ge |x_2|, \\ |x_2| - |y_2| \le |x_2 - y_2| & \text{if } |x_1| < |x_2| \end{cases} \\ &\le |x - y| \end{aligned}$$

which implies that u is Lipschitz continuous. Hence $u \in W^{1,\infty}(Q)$. Since Q is bounded, $u \in W^{1,\infty}(Q) \subset W^{1,p}(Q)$ for any $1 \le p \le \infty$. □

Exercise 5.4 Let $\Omega \subset \mathbb{R}^n$ be open. Given $1 \le p < \infty$, let $u \in W^{1,p}(\Omega)$.

(i) Let $u_+(x) = \max\{u(x), 0\}$ and $u_-(x) = -\min\{u(x), 0\}$. Prove $u_+, u_- \in W^{1,p}(\Omega)$ and show that their weak gradients are given by

$$\nabla u_+(x) = \begin{cases} \nabla u(x) & \text{for almost all } x \text{ with } u(x) > 0, \\ 0 & \text{for almost all } x \text{ with } u(x) \le 0, \end{cases}$$

$$\nabla u_-(x) = \begin{cases} -\nabla u(x) & \text{for almost all } x \text{ with } u(x) < 0, \\ 0 & \text{for almost all } x \text{ with } u(x) \ge 0. \end{cases}$$

(ii) Given $u, v \in W^{1,p}(\Omega)$ and $w(x) = \max\{u(x), v(x)\}$ show that $w \in W^{1,p}(\Omega)$.

(iii) Prove that $\nabla u(x) = 0$ for almost all $x \in \Omega$ with $u(x) = 0$, which means that if $Z = \{x \in \Omega \mid u(x) = 0\}$ and $W = \{x \in \Omega \mid \nabla u(x) = 0 \text{ classically}\}$, then $Z \setminus W$ has Lebesgue measure zero.

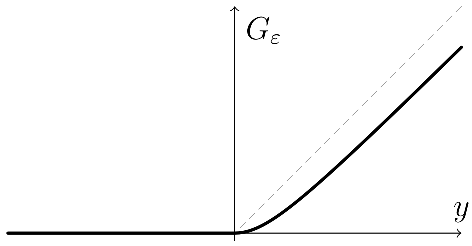
(iv) Let $\lambda \in \mathbb{R}$. Conclude that $\nabla u(x) = 0$ for almost all $x \in \Omega$ with $u(x) = \lambda$.

Solution. Let $\Omega \subset \mathbb{R}^n$ be open. Given $1 \leq p < \infty$, let $u \in W^{1,p}(\Omega)$.

(i) In order to prove $u_+ \in W^{1,p}(\Omega)$, we consider the function $G_\varepsilon \in C^1(\mathbb{R})$ and its derivative G_ε' given by

$$G_\varepsilon(y) = \begin{cases} \sqrt{y^2+\varepsilon^2} - \varepsilon & \text{for } y \geq 0, \\ 0 & \text{for } y < 0, \end{cases}$$

$$G_\varepsilon'(y) = \begin{cases} \frac{y}{\sqrt{y^2+\varepsilon^2}} & \text{for } y \geq 0, \\ 0 & \text{for } y < 0 \end{cases}$$



for some $\varepsilon > 0$. Then, $G_\varepsilon(0) = 0$ and $|G_\varepsilon'| < 1$. By the chain rule, $G_\varepsilon \circ u \in W^{1,p}(\Omega)$ with weak gradient $\nabla(G_\varepsilon \circ u) = (G_\varepsilon' \circ u)\nabla u \in L^p(\Omega)$. Since $|G_\varepsilon \circ u| \leq |u| \in L^p(\Omega)$ and since $(G_\varepsilon \circ u)(x) \to u_+(x)$ as $\varepsilon \to 0$ pointwise almost everywhere, Lebesgue's dominated convergence theorem implies that $\|u_+ - (G_\varepsilon \circ u)\|_{L^p(\Omega)} \to 0$ as $\varepsilon \to 0$. Similarly, $|\nabla(G_\varepsilon \circ u)| = |G_\varepsilon' \circ u||\nabla u| \leq |\nabla u| \in L^p(\Omega)$. If $u(x) > 0$, then $G_\varepsilon'(u(x)) \to 1$ as $\varepsilon \to 0$. Otherwise, $G_\varepsilon'(u(x)) = 0$. Therefore, we have pointwise convergence

$$\nabla(G_\varepsilon \circ u)(x) \xrightarrow{\varepsilon \to 0} g(x) := \begin{cases} \nabla u(x) & \text{for almost all } x \text{ with } u(x) > 0, \\ 0 & \text{for almost all } x \text{ with } u(x) \leq 0 \end{cases}$$

and after application of the dominated convergence theorem, $\|g - \nabla(G_\varepsilon \circ u)\|_{L^p(\Omega)} \to 0$ as $\varepsilon \to 0$. Since the space $W^{1,p}(\Omega)$ is complete, and since $(G_\varepsilon \circ u)$ converges (for a sequence $\varepsilon \to 0$) in $W^{1,p}(\Omega)$, we conclude $u_+ \in W^{1,p}(\Omega)$ with weak gradient $\nabla u_+ = g$. The proof of $u_- \in W^{1,p}(\Omega)$ is identical after replacing $G_\varepsilon(y)$ with $G_\varepsilon(-y)$.

(ii) Let $u, v \in W^{1,p}(\Omega)$. Then, $(u-v)_+ \in W^{1,p}(\Omega)$ by part i. Since

$$w(x) := \max\{u(x), v(x)\} = \max\{u(x) - v(x), 0\} + v(x),$$

we have $w = (u-v)_+ + v \in W^{1,p}(\Omega)$.

(iii) Any $u \in W^{1,p}(\Omega)$ satisfies $u = u_+ - u_-$ with weak gradient $\nabla u = \nabla u_+ - \nabla u_-$. Part (i) implies in particular, that $\nabla u_+(x) = 0$ and $\nabla u_-(x) = 0$ for almost all $x \in \Omega$ with $u(x) = 0$. Consequently, $\nabla u(x) = 0$ for almost all $x \in \Omega$ with $u(x) = 0$.

(iv) Given $\lambda \in \mathbb{R}$ we define $u_\lambda(x) = u(x) - \lambda$. However, unless Ω is bounded, we only have $u_\lambda \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Let $r \geq 1$. Then, $u_\lambda \in W^{1,p}(\Omega \cap B_r)$. By part (iii), $\nabla u(x) = \nabla u_\lambda(x) = 0$ for almost all $x \in \Omega \cap B_r$ with $u_\lambda(x) = 0$. Since a countable union of sets of measure zero still has measure zero and since $\Omega = \bigcup_{r \in \mathbb{N}}(\Omega \cap B_r)$ we conclude that $\nabla u(x) = 0$ for almost all $x \in \Omega$ with $u(x) = \lambda$. $\square$

Exercise 5.5 Let $\alpha \geq 0$. For any $\delta > 0$ we define

$$\mathscr{H}^{\alpha}_{\delta}(A) := \inf\Big\{\sum_{i=1}^{\infty} r_i^{\alpha} \mid A \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i),\ 0 < r_i < \delta,\ x_i \in \mathbb{R}^n\Big\}.$$

The α*-dimensional Hausdorff measure* of any subset $A \subseteq \mathbb{R}^n$ is defined by

$$\mathscr{H}^{\alpha}(A) := \lim_{\delta \searrow 0} \mathscr{H}^{\alpha}_{\delta}(A)$$

Suppose, $K \subset \mathbb{R}^n$ is a compact subset with $\mathscr{H}^{n-\alpha}(K) = 0$ for some $1 \leq \alpha < n$.

(i) For all $1 \leq p \leq \alpha$, prove that K has vanishing $W^{1,p}$-capacity.

(ii) Let $1 \leq p \leq q \leq \infty$ and $\frac{1}{q} + \frac{1}{\alpha} \leq 1$. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u \in L^q(\Omega) \cap C^1(\Omega \setminus K)$ with $|\nabla u| \in L^p(\Omega \setminus K)$. Prove that $u \in W^{1,p}(\Omega)$.

Solution. (i) Let $1 \leq p \leq \alpha$. Let $\varepsilon > 0$. By definition, there exists a collection of balls $\{B_{r_i}(x_i)\}_{i\in\mathbb{N}}$ so that

$$K \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i), \qquad \sum_{i=1}^{\infty} r_i^{n-\alpha} < \varepsilon.$$

Since K is compact, we may suppose that the collection is finite: $i = 1, \ldots, N$. For every $i \in \{1, \ldots, N\}$ there exists a function $\psi_i \in C_c^{\infty}(\mathbb{R}^n)$ satisfying

$$\psi_i = 0 \quad \text{in } \mathbb{R}^n \setminus B_{3r_i}(x_i), \qquad \psi_i = 1 \quad \text{in } B_{2r_i}(x_i), \qquad |\nabla\psi_i| \leq \frac{2}{r_i}.$$

Let $\phi(x) := \max\{\psi_1(x), \ldots, \psi_N(x)\}$. Then, $\phi \in W^{1,p}$ as shown in Exercise 5.4 Moreover, there exists a constant C depending only on n and p such that

$$\int_{\mathbb{R}^n} |\nabla\phi|^p\,dx \leq \sum_{i=1}^{N} \int_{B_{3r_i}(x_i)} |\nabla\psi_i|^p\,dx \leq \sum_{i=1}^{N} C r_i^{n-p} \leq \sum_{i=1}^{N} C r_i^{n-\alpha} < C\varepsilon,$$

where we used $r_i^{-p} \leq r_i^{-\alpha}$ for $p \leq \alpha$ and $r_i < 1$. Let $r_0 := \min\{r_1, \ldots, r_N\}$ and let $0 \leq \rho \in C_c^{\infty}(B_{r_0}(0))$ with $\int_{\mathbb{R}^n} \rho\,dx = 1$. Then the mollification $\varphi := \rho * \phi \in C_c^{\infty}(\mathbb{R}^n)$ has the property that for any $i \in \{1, \ldots, N\}$ and all $x \in B_{r_i}(x_i)$

$$\varphi(x) = \int_{\mathbb{R}^n} \rho(y)\phi(x-y)\,dy = \int_{B_{r_0}(0)} \rho(y)\phi(x-y)\,dy = \int_{B_{r_0}(0)} \rho(y)\,dy = 1,$$

as $|(x-y) - x_i| \leq |x - x_i| + |y| < r_i + r_0 < 2r_i$ for all $x \in B_{r_i}(x_i)$ and all $y \in B_{r_0}(0)$. Hence, $\varphi = 1$ in $\bigcup_{i=1}^{N} B_{r_i}(x_i) \supset K$. Furthermore,

$$\|\nabla\varphi\|_{L^p(\mathbb{R}^n)} = \|\rho * \nabla\phi\|_{L^p(\mathbb{R}^n)} \leq \|\rho\|_{L^1(\mathbb{R}^n)}\|\nabla\phi\|_{L^p(\mathbb{R}^n)} = \|\nabla\phi\|_{L^p(\mathbb{R}^n)} \leq (C\varepsilon)^{\frac{1}{p}}.$$

For every $k \in \mathbb{N}$, let φ_k be the function φ constructed above for the choice $\varepsilon = \frac{1}{k} > 0$. Then $\|\nabla\varphi_k\|_{L^p(\mathbb{R}^n)} \to 0$ as $k \to \infty$. By construction, $\varphi_k(x) \to 0$ for every $x \in \mathbb{R}^n \setminus K$. In particular, $\varphi_k(x) \to 0$ for almost every $x \in \mathbb{R}^n$ because $\mathscr{H}^{n-\alpha}(K) = 0$ implies that K has vanishing Lebesgue measure. Since $\varphi_k = 1$ in a neighbourhood of K, we have shown that K has vanishing $W^{1,p}$-capacity.

(ii) Let $1 \leq p \leq q \leq \infty$ and $\frac{1}{q} + \frac{1}{\alpha} \leq 1$. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u \in L^q(\Omega) \cap C^1(\Omega \setminus K)$ with $|\nabla u| \in L^p(\Omega \setminus K)$. Let $1 \leq s \leq \infty$ such that $\frac{1}{q} + \frac{1}{s} = 1$. Then, $s \leq \alpha$ which by i implies $\operatorname{cap}_{W^{1,s}}(K) = 0$. By Satz 8.1.1, $u \in W^{1,p}(\Omega)$ as claimed.

□

Hints to Exercises.

5.1 Use Parseval's identity: $\|\sum_n a_n \mathrm{e}^{inx}\|^2_{L^2(0,2\pi)} = \frac{1}{2\pi}\sum_n |a_n|^2$ and the Cauchy-Schwarz inequality in the space of ℓ^2 sequences.

5.2 The domain must be nonconvex.

5.3 Who is Rudolf L.?

5.4 For (i), consider the function $G_\varepsilon \circ u$, where $G_\varepsilon \in C^1(\mathbb{R})$ is given by

$$G_\varepsilon(y) = \begin{cases} \sqrt{y^2+\varepsilon^2} - \varepsilon & \text{for } y \geq 0, \\ 0 & \text{for } y < 0. \end{cases}$$

5.5 Use that for any $r > 0$ there exists some $\psi \in C_c^\infty(B_{3r})$ satisfying $\psi = 1$ in B_{2r} and $|\nabla\psi| \leq \frac{2}{r}$.